Formal Poincaré-Dulac renormalization for holomorphic germs

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Abstract. We describe a general renormalization procedure for germs of holomorphic (or even formal) self-maps, producing a formal normal form simpler than the classical Poincaré-Dulac normal form. As an example of application of our method we provide a complete list of normal forms for quadratic bi-dimensional superattracting germs, that could not be simplified using the classical Poincaré-Dulac normalization only. Finally, we also discuss a few examples of renormalization of germs tangent to the identity, revealing interesting second-order resonance phenomena.

0. Introduction

In the study of a class of holomorphic dynamical systems, an important goal often is the classification under topological, holomorphic or formal conjugation. In particular, for each dynamical system in the class one would like to have a definite way of choosing a (hopefully simpler, possibly unique) representative in the same conjugacy class; a *normal form* of the original dynamical system.

The most famous kind of normal form for local holomorphic dynamical systems (i.e., germs of holomorphic vector fields at a singular point, or germs of holomorphic self-maps with a fixed point) is the *Poincaré-Dulac normal form* with respect to formal conjugation, introduced at the end of the nineteenth century. Let us recall very quickly its definition, at least in the setting we are interested here, that is of germs of self-maps with a fixed point, that we can assume to be the origin in $\mathbb{C}^n$. Moreover, since we are discussing formal normal forms, we shall work with formal transformations of $\mathbb{C}^n$, that is n-tuples of power series, without discussing here convergence issues.

So let $F \in \widehat{\mathcal{O}}^n$ be a formal transformation in n complex variables, where $\widehat{\mathcal{O}}^n$ denotes the space of n-tuples of power series in n variables fixing the origin (that is, with vanishing constant term), and let Λ denote the (not necessarily invertible) linear term of F. For simplicity, given a linear map $\Lambda \in M_{n,n}(\mathbb{C})$ we shall denote by $\widehat{\mathcal{O}}^n_\Lambda$ the set of formal transformations in $\widehat{\mathcal{O}}^n$ with Λ as linear part. If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of Λ, we shall say that a multi-index $Q = (q_1, \ldots, q_n) \in \mathbb{N}^n$ with $q_1 + \cdots + q_n \geq 2$ is Λ-*resonant* if there is $j \in \{1, \ldots, n\}$ such that $\lambda_1^{q_1} \cdots \lambda_n^{q_n} = \lambda_j$. If this happens, we shall say that the monomial $z_1^{q_1} \cdots z_n^{q_n} e_j$ is Λ-*resonant*, where $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{C}^n$. Then (for a proof see, e.g., [Ar]):

Theorem 0.1: (Poincaré 1893, Dulac 1904) *Let $F \in \widehat{\mathcal{O}}^n_\Lambda$ be a formal transformation in n complex variables fixing the origin, with Λ in Jordan normal form. Then there exists an invertible formal transformation $\Phi \in \widehat{\mathcal{O}}^n_I$ with identity linear part such that $G = \Phi^{-1} \circ F \circ \Phi$ contains only Λ-resonant monomials.*

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The formal transformation G is a *Poincaré-Dulac normal form* of F; notice that, since $\Phi \in \widehat{\mathcal{O}}^n_I$, the linear part of G is still Λ. More generally, we shall say that a $G \in \widehat{\mathcal{O}}^n_\Lambda$ is *in Poincaré-Dulac normal form* if G contains only Λ-resonant monomials.

The importance of this result cannot be underestimated, and it has been applied uncountably many times; however it has some limitations. For instance, if $\Lambda = O$ or $\Lambda = I$ then *all* monomials are resonant; and thus in these cases the Poincaré-Dulac normal form reduces to the original map. More generally, as we shall try and explain below, even when the Poincaré-Dulac normal form is a simplification of the original germ, it is still possible to further simplify the germ (to *renormalize* it) by applying invertible transformations preserving the property of being in Poincaré-Dulac normal form.

This idea of renormalizing Poincaré-Dulac normal forms is not new in the context of vector fields, where it also has been studied the concept of hypernormal forms, obtained (roughly speaking) by renormalizing infinitely many times; see, e.g., [AFGG, B, BS, G, KOW, LS, Mu1, Mu2] and references therein. On the other hand, this idea has not yet been fully exploited in the context of self-maps (one of the few exceptions is [AT1], where it is applied to a particular class of self-maps with identity linear part). The aim of this paper is to describe in general the renormalization procedure for formal transformations in several complex variables, following the general ideas (but with significantly different details) of the vector field case. We shall then apply this procedure to the case of superattracting (i.e., with $\Lambda = O$) 2-dimensional formal transformations, case that has no analogue in the vector field setting. We shall also discuss a few interesting examples with $\Lambda = I$, in particularly showing the appearance of second-order resonance phenomena.

We conclude this introduction by roughly describing the renormalization procedure. To explain it better, let us first recast the Poincaré-Dulac normalization in slightly different terms (see also [Rü]). For each $\nu \geq 2$ let $\mathcal{H}^\nu$ denote the space of n-tuples of homogeneous polynomials in n variables of degree ν. Then every $F \in \widehat{\mathcal{O}}^n_\Lambda$ admits a *homogeneous expansion*

$$F = \Lambda + \sum_{\nu \geq 2} F_\nu \ ,$$

where $F_\nu \in \mathcal{H}^\nu$ is the *ν-homogeneous term* of F. We shall also use the notation $\{G\}_\nu$ to denote the ν-homogeneous term of a formal transformation G.

If

$$\Phi = I + \sum_{\nu \geq 2} H_\nu$$

is the homogeneous expansion of an invertible formal transformation $\Phi \in \widehat{\mathcal{O}}^n_I$, then it turns out that there exists a linear operator $L_\Lambda \colon \widehat{\mathcal{O}}^n \to \widehat{\mathcal{O}}^n$, given by

$$L_\Lambda(H) = H \circ \Lambda - \Lambda H \ ,$$

sending each $\mathcal{H}^\nu$ into itself and such that

$$\{\Phi^{-1} \circ F \circ \Phi\}_\nu = F_\nu - L_\Lambda(H_\nu) + R_\nu$$

for all $\nu \geq 2$, where R_ν is a remainder term depending only on F_ρ and H_σ with ρ, $\sigma < \nu$. This suggests to consider for each $\nu \geq 2$ splittings of the form

$$\mathcal{H}^\nu = \operatorname{Im} L_\Lambda|_{\mathcal{H}^\nu} \oplus \mathcal{N}^\nu \quad \text{and} \quad \mathcal{H}^\nu = \operatorname{Ker} L_\Lambda|_{\mathcal{H}^\nu} \oplus \mathcal{M}^\nu$$

where $\mathcal{N}^\nu$ and $\mathcal{M}^\nu$ are suitable complementary subspaces. Then, arguing by induction (see Proposition 2.4), for each $\nu \geq 2$ it is possible to find a unique $H_\nu \in \mathcal{M}^\nu$ such that

$$\{\Phi^{-1} \circ F \circ \Phi\}_\nu = F_\nu - L_\Lambda(H_\nu) + R_\nu \in \mathcal{N}^\nu \ ; \tag{0.1}$$

we can then say that $G = \Phi^{-1} \circ F \circ \Phi$ is in *first order normal form* with respect to the given choice of complementary subspaces.

When Λ is diagonal, $\operatorname{Ker} L_\Lambda$ is generated by the resonant monomials, and $\operatorname{Im} L_\Lambda$ is generated by the non-resonant monomials; in particular, for each $\nu \geq 2$ we have the splitting

$$\mathcal{H}^\nu = \operatorname{Im} L_\Lambda|_{\mathcal{H}^\nu} \oplus \operatorname{Ker} L_\Lambda|_{\mathcal{H}^\nu} \ .$$

Thus taking $\mathcal{N}^\mu = \operatorname{Ker} L_\Lambda|_{\mathcal{H}^\nu}$ and $\mathcal{M}^\mu = \operatorname{Im} L_\Lambda|_{\mathcal{H}^\nu}$ we see that the classical Poincaré-Dulac normal form coincides with the first order normal form with respect to these complementary subspaces (when Λ has a nilpotent part the situation is only slightly more complicated; see Section 2 for details).

A consequence of (0.1) is that if $H_\nu \in \operatorname{Ker} L_\Lambda$ then $\{\Phi^{-1} \circ F \circ \Phi\}_\nu$ does not depend on H_ν; however, H_ν does affect the remainder terms R_ρ with $\rho > \nu$. In other words, we can use the terms $H_\sigma \in \operatorname{Ker} L_\Lambda$ with $\sigma < \nu$ to simplify the remainder term R_ν.

More precisely, if we take F is in first order normal form, that is

$$F = \Lambda + \sum_{\nu \geq \mu} F_\nu$$

with $F_\nu \in \mathcal{N}^\nu$ for all $\nu \geq \mu$ (and $F_\mu \neq O$), and take $\Phi \in \widehat{\mathcal{O}}^n_I$ such that $H_\nu \in \operatorname{Ker} L_\Lambda$ for all $\nu \geq 2$, it turns out that there is an operator $L_{F_\mu,\Lambda}\colon \widehat{\mathcal{O}}^n \to \widehat{\mathcal{O}}^n$ sending each $\mathcal{H}^{\nu-\mu+1}$ in $\mathcal{H}^\nu$ such that

$$\{\Phi^{-1} \circ F \circ \Phi\}_\nu = F_\nu - L_{F_\mu,\Lambda}(H_{\nu-\mu+1}) + R'_\nu$$

for all $\nu \geq \mu$, where R'_ν is a remainder term depending only on F_ρ with $\rho < \nu$ and on H_σ with $\sigma < \nu - \mu + 1$ (see Theorem 2.3 for a more complete formula valid without assumptions on F and Φ). The operator $L_{F_\mu,\Lambda}$ is given by

$$L_{F_\mu,\Lambda}(H) = \big((\operatorname{Jac} H) \circ \Lambda\big) \cdot F_\mu - (\operatorname{Jac} F_\mu) \cdot H \ ;$$

notice that, contrarily to L_Λ, the operator $L_{F_\mu,\Lambda}$ is different from the operators appearing in the renormalization or hypernormalization of singular vector fields, and thus it has to be studied on its own.

If the subspaces $\mathcal{N}^\nu$ are chosen (as will be in our case when Λ is diagonalizable) so that

$$L_{F_\mu,\Lambda}(\operatorname{Ker} L_\Lambda \cap \mathcal{H}^{\nu-\mu+1}) \subseteq \mathcal{N}^\nu \tag{0.2}$$

for all $\nu \geq \mu$ (notice that this condition is particularly easy to state if $\mathcal{N}^\nu = \operatorname{Ker} L_\Lambda \cap \mathcal{H}^\nu$), then $R'_\nu \in \mathcal{N}^\mu$ for all $\nu \geq \mu$. Therefore we can argue as before: putting, for simplicity, $\mathcal{H}^\nu_\Lambda = \operatorname{Ker} L_\Lambda \cap \mathcal{H}^\nu$, if we choose splittings

$$\mathcal{N}^\nu = \operatorname{Im} L_{F_\mu,\Lambda}|_{\mathcal{H}^{\nu-\mu+1}_\Lambda} \oplus \tilde{\mathcal{N}}^\nu$$

and

$$\mathcal{H}^{\nu-\mu+1}_\Lambda = \operatorname{Ker} L_{F_\mu,\Lambda}|_{\mathcal{H}^{\nu-\mu+1}_\Lambda} \oplus \tilde{\mathcal{M}}^{\nu-\mu+1}$$

then arguing again by induction for each $\nu \geq \mu$ it is possible to find a unique $H_{\nu-\mu+1} \in \tilde{\mathcal{M}}^{\nu-\mu+1}$ such that

$$\{\Phi^{-1} \circ F \circ \Phi\}_\nu = F_\nu - L_{F_\mu,\Lambda}(H_{\nu-\mu+1}) + R'_\nu \in \tilde{\mathcal{N}}^\nu \ .$$

We shall then say that $G = \Phi^{-1} \circ F \circ \Phi$ is in *renormalized* (or *second order*) *Poincaré-Dulac normal form*, with respect to the chosen complementary subspaces.

We are left with saying how to choose the complementary subspaces. In this paper, we shall use the orthogonal subspaces with respect to the Fischer Hermitian product, defined by

$$\langle z_1^{p_1} \cdots z_n^{p_n} e_h, z_1^{q_1} \cdots z_n^{q_n} e_k \rangle = \begin{cases} 0 & \text{if } h \neq k \text{ or } p_j \neq q_j \text{ for some } j; \\ \dfrac{p_1! \cdots p_n!}{(p_1 + \cdots + p_n)!} & \text{if } h = k \text{ and } p_j = q_j \text{ for all } j. \end{cases} \tag{0.3}$$

The reason of this choice is that, as we shall see in Sections 3 and 4, it will substantially simplify the expression of the renormalized Poincaré-Dulac normal forms. In particular, when $\Lambda = O$ and $n = 2$, it turns out that, except in a few degenerate cases, the renormalized Poincaré-Dulac normal forms depend on two power series of *one* variable only.

1. Homogeneous maps

In this section we shall collect a few results on homogeneous polynomials and maps we shall need later.

Definition 1.1: We shall denote by $\mathcal{H}^d = \big(\mathbb{C}_d[z]\big)^n$ the space of *homogenous maps of degree* d, i.e., of n-tuples of homogeneous polynomials of degree $d \geq 1$ in the variables $(z_1, \ldots, z_n)$. It is well known (see, e.g., [Car, pp. 79–88]) that to each $P \in \mathcal{H}^d$ is associated a *unique* symmetric multilinear map $\tilde{P}\colon (\mathbb{C}^n)^d \to \mathbb{C}^n$ such that

$$P(z) = \tilde{P}(z, \ldots, z)$$

for all $z \in \mathbb{C}^n$. Notice that $\widehat{\mathcal{O}}^n = \prod\limits_{d \geq 1} \mathcal{H}^d$; we set $\mathcal{H} = \prod\limits_{d \geq 2} \mathcal{H}^d$.

Roughly speaking, the symmetric multilinear map associated to a homogeneous map H encodes the derivatives of H. For instance, we have

Lemma 1.1: *If $H \in \mathcal{H}^d$ we have*

$$(\mathrm{Jac}\, H)(z) \cdot v = d\, \tilde{H}(v, z, \ldots, z)$$

for all z, $v \in \mathbb{C}^n$.

Proof: For $j = 1, \ldots, n$ and $z \in \mathbb{C}^n$ we have

$$\frac{H(z + he_j) - H(z)}{h} = \frac{\tilde{H}(z + he_j, \ldots, z + he_j) - \tilde{H}(z, \ldots, z)}{h} = d\, \tilde{H}(e_j, z, \ldots, z) + O(h) \ ,$$

where e_j is the j-th vector of the canonical basis of $\mathbb{C}^n$. Therefore $\frac{\partial H}{\partial z_j}(z) = d\, \tilde{H}(e_j, z, \ldots, z)$ and

$$(\mathrm{Jac}\, H)(z) \cdot v = \sum_{j=1}^n \frac{\partial H}{\partial z_j}(z) v_j = d \sum_{j=1}^n \tilde{H}(e_j, z, \ldots, z) v_j = d\, \tilde{H}(v, z, \ldots, z) \ .$$

□

Later on we shall need to compute the multilinear map associated to a homogeneous map obtained as a composition. The formula we are interested in is contained in the next lemma.

Lemma 1.2: *Assume that $P \in \mathcal{H}^d$ is of the form*

$$P(z) = \tilde{K}\big(H_{d_1}(z), \ldots, H_{d_r}(z)\big) ,$$

where $\tilde{K}$ is r-multilinear, $d_1 + \cdots + d_r = d$, and $H_{d_j} \in \mathcal{H}^{d_j}$ for $j = 1, \ldots, r$. Then

$$\tilde{P}(v, w, \ldots, w) = \frac{1}{d} \sum_{j=1}^{r} d_j \tilde{K}\big(H_{d_1}(w), \ldots, \tilde{H}_{d_j}(v, w, \ldots, w), \ldots, H_{d_r}(w)\big)$$

for all v, $w \in \mathbb{C}^n$.

Proof: Write $z = w + \varepsilon v$. Then

$$\begin{aligned} &P(w) + d\varepsilon \tilde{P}(v, w, \ldots, w) + O(\varepsilon^2) \\ &\quad = P(w + \varepsilon v) = \tilde{K}\big(\tilde{H}_{d_1}(w + \varepsilon v, \ldots, w + \varepsilon v), \ldots, \tilde{H}_{d_r}(w + \varepsilon v, \ldots, w + \varepsilon v)\big) \\ &\quad = \tilde{K}\big(H_{d_1}(w), \ldots, H_{d_r}(w)\big) + \varepsilon \sum_{j=1}^{r} d_j \tilde{K}\big(H_{d_1}(w), \ldots, \tilde{H}_{d_j}(v, w, \ldots, w), \ldots, H_{d_r}(w)\big) + O(\varepsilon^2) , \end{aligned}$$

and we are done. □

Definition 1.2: Given a linear map $\Lambda \in M_{n,n}(\mathbb{C})$, we define a linear operator $L_\Lambda \colon \mathcal{H} \to \mathcal{H}$ by setting

$$L_\Lambda(H) = H \circ \Lambda - \Lambda H .$$

We shall say that a homogeneous map $H \in \mathcal{H}^d$ is Λ-*resonant* if $L_\Lambda(H) = O$, and we shall denote by $\mathcal{H}^d_\Lambda = \operatorname{Ker} L_\Lambda \cap \mathcal{H}^d$ the subspace of Λ-resonant homogeneous maps of degree d. Finally, we set $\mathcal{H}_\Lambda = \prod_{d \geq 2} \mathcal{H}^d_\Lambda$.

When Λ is diagonal, then the Λ-resonant monomials are exactly the resonant monomials appearing in the classical Poincaré-Dulac theory.

Definition 1.3: If $Q = (q_1, \ldots, q_n) \in \mathbb{N}^n$ is a multi-index and $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, we shall put $z^Q = z_1^{q_1} \cdots z_n^{q_n}$. Given $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n) \in M_{n,n}(\mathbb{C})$, we shall say that $Q \in \mathbb{N}^n$ with $q_1 + \cdots + q_n \geq 2$ is Λ-*resonant* on the j-th coordinate if $\lambda_1^{q_1} \cdots \lambda_n^{q_n} = \lambda_j$. We shall denote by $\operatorname{Res}_j(\Lambda)$ the set of multi-indices Λ-resonant on the j-th coordinate.

Remark 1.1: If $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n) \in M_{n,n}(\mathbb{C})$ is diagonal, and $z^Q e_j \in \mathcal{H}^d$ is a homogeneous monomial (with $q_1 + \cdots + q_n = d$), then (identifying the matrix Λ with the vector, still denoted by Λ, of its diagonal entries) we have

$$L_\Lambda(z^Q e_j) = (\Lambda^Q - \lambda_j) z^Q e_j .$$

Therefore $z^Q e_j$ is Λ-resonant if and only if Q is Λ-resonant in the j-th coordinate, that is if and only if $Q \in \operatorname{Res}_j(\Lambda)$. In particular, a basis of $\mathcal{H}^d_\Lambda$ is given by $z^Q e_j$ with $Q \in \operatorname{Res}_j(\Lambda)$ and $q_1 + \cdots + q_n = d$, and we have

$$\mathcal{H}^d = \mathcal{H}^d_\Lambda \oplus \operatorname{Im} L_\Lambda|_{\mathcal{H}^d}$$

for all $d \geq 2$.

It is possible to detect the Λ-resonance by using the associated multilinear map:

Lemma 1.3: *If $\Lambda\in M_{n,n}(\mathbb{C})$ and $H\in\mathcal{H}^d$ then H is Λ-resonant if and only if*

$$\tilde H(\Lambda v_1,\dots,\Lambda v_d)=\Lambda\tilde H(v_1,\dots,v_d) \tag{1.1}$$

for all $v_1,\dots,v_d\in\mathbb{C}^n$. In particular, if $H\in\mathcal{H}^d_\Lambda$ then

$$\bigl((\mathrm{Jac}\,H)\circ\Lambda\bigr)\cdot\Lambda=\Lambda\cdot(\mathrm{Jac}\,H)\;. \tag{1.2}$$

Proof: One direction is trivial. Conversely, assume $H\in\mathcal{H}^d_\Lambda$. By definition, H is Λ-resonant if and only if $\tilde H(\Lambda w,\dots,\Lambda w)=\Lambda\tilde H(w,\dots,w)$ for all $w\in\mathbb{C}^n$. Put $w=z+\varepsilon v_1$; then

$$\begin{aligned}\tilde H(\Lambda z,\dots,\Lambda z)+\varepsilon d\,\tilde H(\Lambda v_1,\Lambda z,\dots,\Lambda z)+O(\varepsilon^2)&=\tilde H\bigl(\Lambda(z+\varepsilon v_1),\dots,\Lambda(z+\varepsilon v_1)\bigr)\\&=\Lambda\tilde H(z+\varepsilon v_1,\dots,z+\varepsilon v_1)\\&=\Lambda\tilde H(z,\dots,z)+\varepsilon d\,\Lambda\tilde H(v_1,z,\dots,z)+O(\varepsilon^2)\;,\end{aligned}$$

and thus

$$\tilde H(\Lambda v_1,\Lambda z,\dots,\Lambda z)=\Lambda\tilde H(v_1,z,\dots,z)\;; \tag{1.3}$$

in particular (1.2) is a consequence of Lemma 1.1.

Now put $z=z_1+\varepsilon v_2$ in (1.3). We get

$$\begin{aligned}\tilde H(\Lambda v_1,\Lambda z_1,\dots,\Lambda z_1)&+\varepsilon(d-1)\tilde H(\Lambda v_1,\Lambda v_2,\Lambda z_1,\dots,\Lambda z_1)+O(\varepsilon^2)\\&=\tilde H\bigl(\Lambda v_1,\Lambda(z_1+\varepsilon v_2),\dots,\Lambda(z_1+\varepsilon v_2)\bigr)\\&=\Lambda\tilde H(v_1,z_1+\varepsilon v_2,\dots,z_1+\varepsilon v_2)\\&=\Lambda\tilde H(v_1,z_1,\dots,z_1)+\varepsilon(d-1)\Lambda\tilde H(v_1,v_2,z,\dots,z)+O(\varepsilon^2)\;,\end{aligned}$$

and hence

$$\tilde H(\Lambda v_1,\Lambda v_2,\Lambda z_1,\dots,\Lambda z_1)=\Lambda\tilde H(v_1,v_2,z_1,\dots,z_1)$$

for all v_1, v_2, $z_1\in\mathbb{C}^n$. Proceeding in this way we get (1.1). □

As we shall see in the next section, the operator L_Λ appears in the usual Poincaré-Dulac normalization; for the renormalization we shall need a different operator, that we now introduce.

Definition 1.4: Given $P\in\mathcal{H}^\mu$ and $\Lambda\in M_{n,n}(\mathbb{C})$, let $L_{P,\Lambda}\colon\mathcal{H}^d\to\mathcal{H}^{d+\mu-1}$ be given by

$$L_{P,\Lambda}(H)(z)=d\,\tilde H\bigl(P(z),\Lambda z,\dots,\Lambda z\bigr)-\mu\tilde P\bigl(H(z),z,\dots,z\bigr)\;.$$

Remark 1.2: Lemma 1.1 implies that

$$d\,\tilde H\bigl(P(z),\Lambda z,\dots,\Lambda z\bigr)=(\mathrm{Jac}\,H)(\Lambda z)\cdot P(z)\;.$$

Therefore

$$L_{P,\Lambda}(H)=\bigl((\mathrm{Jac}\,H)\circ\Lambda\bigr)\cdot P-(\mathrm{Jac}\,P)\cdot H\;.$$

In particular, when $\Lambda=O$ we have

$$L_{P,O}(H)=-(\mathrm{Jac}\,P)\cdot H\;.$$

Remark 1.3: If we take $\mu=1$ and $P=\Lambda$ we find

$$L_{\Lambda,\Lambda}(H)=d\tilde H(\Lambda,\dots,\Lambda)-\Lambda H=(d-1)H\circ\Lambda+L_\Lambda(H)\;;$$

in particular, $L_{\Lambda,\Lambda}\neq L_\Lambda$.

As mentioned in the introduction, for our machinery to work is important that the operator $L_{P,\Lambda}$ sends the kernel of L_Λ into itself. This is the last result of this section:

Corollary 1.4: *Take $\Lambda\in M_{n,n}(\mathbb{C})$ and $P\in\mathcal{H}^\mu_\Lambda$. Then $L_{P,\Lambda}(\mathcal{H}^d_\Lambda)\subseteq\mathcal{H}^{d+\mu-1}_\Lambda$ for all $d\ge 2$.*

Proof: Using Lemma 1.3 and the definition of $L_{P,\Lambda}$, if $H\in\mathcal{H}^d_\Lambda$ we get

$$\begin{aligned}L_{P,\Lambda}(H)(\Lambda z)&=d\,\tilde H\bigl(P(\Lambda z),\Lambda^2z,\dots,\Lambda^2z\bigr)-\mu\tilde P\bigl(H(\Lambda z),\Lambda z,\dots,\Lambda z\bigr)\\&=d\,\tilde H\bigl(\Lambda P(z),\Lambda^2z,\dots,\Lambda^2z\bigr)-\mu\tilde P\bigl(\Lambda H(z),\Lambda z,\dots,\Lambda z\bigr)\\&=d\,\Lambda\tilde H\bigl(P(z),\Lambda z,\dots,\Lambda z\bigr)-\mu\Lambda\tilde P\bigl(H(z),z,\dots,z\bigr)\\&=\Lambda L_{P,\Lambda}(H)(z)\;.\end{aligned}$$

□

2. Renormalization

The aim of this section is to describe a procedure associating a renormalized formal Poincaré-Dulac normal form to any germ of holomorphic (or even formal) self-map of $\mathbb{C}^n$ fixing the origin. This is particularly interesting in the case of germs superattracting (i.e., with vanishing linear part) or tangent to the identity (i.e., with identity linear part), because in those cases all monomials are resonant and so the usual Poincaré-Dulac procedure just gives back the original germ.

The idea is the following. Studying the classical proof (see, e.g., [Ar], [R1, 2] and Proposition 2.4 below) of the standard Poincaré-Dulac normalization of a germ F, it is clear that we find a unique formal germ Ψ tangent to the identity and containing only *nonresonant* monomials such that $F_1=\Psi^{-1}\circ F\circ\Psi$ is in Poincaré-Dulac normal form, that is contains only resonant monomials. Choosing suitable positive definite Hermitian products on the spaces of homogeneous polynomial maps, we shall then be able to determine a *unique* formal germ Φ tangent to the identity and containing only *resonant* monomials such that $\Phi^{-1}\circ F_1\circ\Phi$ is (in a precise sense) a renormalized Poincaré-Dulac normal form of F.

Let us fix a few definitions and notations.

Definition 2.1: We shall denote by $\widehat{\mathcal{O}}^n$ the space of n-tuples of formal power series with vanishing constant term. Furthermore, given $\Lambda\in M_{n,n}(\mathbb{C})$ we shall denote by $\widehat{\mathcal{O}}^n_\Lambda$ the subset of $F\in\widehat{\mathcal{O}}^n$ with $dF_O=\Lambda$.

Definition 2.2: Every $F\in\widehat{\mathcal{O}}^n$ can be written in a unique way as a formal sum

$$F=\sum_{d\ge1}F_d \tag{2.1}$$

with $F_d\in\mathcal{H}^d$; (2.1) is the *homogeneous expansion* of F, and F_d is the *d-homogeneous term* of F. We shall often write $\{F\}_d$ for F_d. In particular, if $F\in\widehat{\mathcal{O}}^n_\Lambda$ then $\{F\}_1=\Lambda$.

The homogeneous terms behave in a predictable way with respect to composition and inverse:

Lemma 2.1: *Take F, $G\in\widehat{\mathcal{O}}^n_O$, and let $F=\sum\limits_{d\ge1}F_d$ and $G=\sum\limits_{d\ge1}G_d$ be their homogeneous expansions. Then*

$$\{F\circ G\}_d=\sum_{\substack{1\le r\le d\\ d_1+\cdots+d_r=d}}\tilde F_r(G_{d_1},\dots,G_{d_r})$$

for all $d\ge1$.

Proof: See, e.g., [LS, Lemma A.4]. □

Lemma 2.2: *Take $\Phi \in \widehat{\mathcal{O}}^n_I$ with homogeneous expansion $\Phi = I + \sum\limits_{d\ge\delta} H_d$, for some $\delta \ge 2$, and let $\Phi^{-1} = I + \sum\limits_{d\ge 2} K_d$ be the homogeneous expansion of the inverse. Then*

$$K_d = -H_d - \sum_{\substack{2\le r\le d-1\\ d_1+\cdots+d_r=d}} \tilde K_r(H_{d_1},\ldots,H_{d_r}) \tag{2.2}$$

for all $d \ge 2$. In particular, $K_\delta = -H_\delta$, and $K_d = O$ for $d = 2,\ldots,\delta-1$. Furthermore, if $H_2,\ldots,H_d$ are Λ-resonant for some $\Lambda \in M_{n,n}(\mathbb{C})$ then also K_d is.

Proof: Lemma 2.1 yields

$$\sum_{\substack{1\le r\le d\\ d_1+\cdots+d_r=d}} \tilde K_r(H_{d_1},\ldots,H_{d_r}) = \{\Phi^{-1}\circ\Phi\}_d = O \tag{2.3}$$

for all $d \ge 2$. Now, when $r = 1$ we necessarily have $d_1 = d$, and so $\tilde K_1(H_d) = H_d$ because $K_1 = I$. Analogously, when $r = d$ we necessarily have $d_1 = \cdots = d_r = 1$, and so $\tilde K_d(H_1,\ldots,H_1) = K_d$ because $H_1 = I$. Therefore (2.3) becomes

$$H_d + \sum_{\substack{2\le r\le d-1\\ d_1+\cdots+d_r=d}} \tilde K_r(H_{d_1},\ldots,H_{d_r}) + K_d = O\ ,$$

and (2.2) follows. In particular, if $2 \le d \le \delta$ in the sum in (2.2) we have $2 \le d_j < \delta$ (and hence $H_{d_j} = O$) for at least one $j = 1,\ldots,r$; thus $K_d = -H_d$ for $2 \le d \le \delta$, as claimed.

Finally, to prove the last assertion we argue by induction. Assume that $H_2,\ldots,H_d$ are Λ-resonant. If $d = \delta$ then $K_\delta = -H_\delta$ and thus K_δ is clearly Λ-resonant. Assume the assertion true for $d-1$; in particular, $K_\delta,\ldots,K_{d-1}$ are Λ-resonant. Then

$$\begin{aligned} K_d\circ\Lambda &= -H_d\circ\Lambda - \sum_{\substack{2\le r\le d-1\\ d_1+\cdots+d_r=d}} \tilde K_r(H_{d_1}\circ\Lambda,\ldots,H_{d_r}\circ\Lambda)\\ &= \Lambda H_d - \sum_{\substack{2\le r\le d-1\\ d_1+\cdots+d_r=d}} \tilde K_r(\Lambda H_{d_1},\ldots,\Lambda H_{d_r}) = \Lambda K_d \end{aligned}$$

because $K_2,\ldots,K_{d-1}$ are Λ-resonant (and we are using Lemma 1.3). □

Definition 2.3: Given $\Lambda \in M_{n,n}(\mathbb{C})$, we shall say that $F \in \widehat{\mathcal{O}}^n$ is Λ-*resonant* if $F\circ\Lambda = \Lambda F$. Clearly, F is Λ-resonant if and only if $\{F\}_d \in \mathcal{H}^d_\Lambda$ for all $d \in \mathbb{N}$.

The main technical result of this section is the following:

Theorem 2.3: *Given $F \in \widehat{\mathcal{O}}^n_O$, let $F = \Lambda + \sum\limits_{d\ge\mu} F_d$ be its homogeneous expansion, with $F_\mu \ne O$. Then for every $\Phi \in \widehat{\mathcal{O}}^n_I$ with homogeneous expansion $\Phi = I + \sum\limits_{d\ge 2} H_d$ and every $\nu \ge 2$ we have*

$$\{\Phi^{-1}\circ F\circ\Phi\}_\nu = F_\nu - L_\Lambda(H_\nu) - L_{F_\mu,\Lambda}(H_{\nu-\mu+1}) + Q_\nu + R_\nu\ , \tag{2.4}$$

where Q_ν depends only on Λ and on H_γ with $\gamma < \nu$, while R_ν depends only on F_ρ with $\rho < \nu$ and on H_γ with $\gamma < \nu - \mu + 1$, and we put $L_{F_\mu,\Lambda}(H_1) = O$. Furthermore, we have:

(i) *if* $H_2,\dots,H_{\nu-1}\in\mathcal{H}_\Lambda$ *then* $Q_\nu=O$*; in particular, if* Φ *is* Λ*-resonant then* $L_\Lambda(H_\nu)=Q_\nu=O$ *for all* $\nu\ge 2$*;*

(ii) *if* Φ *is* Λ*-resonant then* $\{\Phi^{-1}\circ F\circ\Phi\}_\nu=O$ *for* $2\le\nu<\mu$, $\{\Phi^{-1}\circ F\circ\Phi\}_\mu=F_\mu$*, and*

$$\{\Phi^{-1}\circ F\circ\Phi\}_{\mu+1}=F_{\mu+1}-L_{F_\mu,\Lambda}(H_2)\ ;$$

(iii) *if* $F=\Lambda$ *then* $R_\nu=O$ *for all* $\nu\ge 2$*;*

(iv) *if* $F_2,\dots,F_{\nu-1}$ *and* $H_2,\dots,H_{\nu-\mu}$ *are* Λ*-resonant then* R_ν *is* Λ*-resonant.*

Proof: Using twice Lemma 2.1 we get

$$\begin{aligned}\{\Phi^{-1}\circ F\circ\Phi\}_\nu&=\sum_{\substack{1\le s\le\nu\\ \nu_1+\cdots+\nu_s=\nu}}\tilde K_s(\{F\circ\Phi\}_{\nu_1},\dots,\{F\circ\Phi\}_{\nu_s})\\
&=\sum_{\substack{1\le s\le\nu\\ \nu_1+\cdots+\nu_s=\nu}}\ \sum_{\substack{1\le r_1\le\nu_1\\ d_{11}+\cdots+d_{1r_1}=\nu_1}}\cdots\sum_{\substack{1\le r_s\le\nu_s\\ d_{s1}+\cdots+d_{sr_s}=\nu_s}}\tilde K_s\bigl(\tilde F_{r_1}(H_{d_{11}},\dots,H_{d_{1r_1}}),\dots,\tilde F_{r_s}(H_{d_{s1}},\dots,H_{d_{sr_s}})\bigr)\\
&=T_\nu+S_1(\nu)+\sum_{s\ge 2}S_s(\nu)\ ,\end{aligned}$$

where $\Phi^{-1}=I+\sum\limits_{d\ge 2}K_d$ is the homogeneous expansion of Φ^{-1}, and:

(1)
$$T_\nu=\sum_{\substack{1\le s\le\nu\\ \nu_1+\cdots+\nu_s=\nu}}\tilde K_s(\Lambda H_{\nu_1},\dots,\Lambda H_{\nu_s})$$

is obtained considering only the terms with $r_1=\ldots=r_s=1$;

(2)
$$S_1(\nu)=\sum_{\substack{\mu\le r\le\nu\\ d_1+\cdots+d_r=\nu}}\tilde F_r(H_{d_1},\dots,H_{d_r})$$

contains the terms with $s=1$ and $r_1>1$; and

(3)
$$S_s(\nu)=\sum_{\nu_1+\cdots+\nu_s=\nu}\ \sum_{\substack{1\le r_1\le\nu_1\\ \vdots\\ 1\le r_s\le\nu_s\\ \max\{r_1,\dots,r_s\}\ge\mu}}\ \sum_{\substack{d_{11}+\cdots+d_{1r_1}=\nu_1\\ \vdots\\ d_{s1}+\cdots+d_{sr_s}=\nu_s}}\tilde K_s\bigl(\tilde F_{r_1}(H_{d_{11}},\dots,H_{d_{1r_1}}),\dots,\tilde F_{r_s}(H_{d_{s1}},\dots,H_{d_{sr_s}})\bigr)$$

contains the terms with fixed $s\ge 2$ and at least one r_j greater than 1 (and thus greater than or equal to μ, because $F_2=\ldots=F_{\mu-1}=O$ by assumption).

Let us first study T_ν. The summand corresponding to $s=1$ is ΛH_ν; the summand corresponding to $s=\nu$ is $K_\nu\circ\Lambda$; therefore

$$T_\nu=\Lambda H_\nu+K_\nu\circ\Lambda+\sum_{\substack{2\le s\le\nu-1\\ \nu_1+\cdots+\nu_s=\nu}}\tilde K_s(\Lambda H_{\nu_1},\dots,\Lambda H_{\nu_s})=-L_\Lambda(H_\nu)+Q_\nu\ ,$$

where, using Lemma 2.2 to express K_ν,

$$Q_\nu=\sum_{\substack{2\le s\le\nu-1\\ \nu_1+\cdots+\nu_s=\nu}}\Bigl[\tilde K_s(\Lambda H_{\nu_1},\dots,\Lambda H_{\nu_s})-\tilde K_s(H_{\nu_1}\circ\Lambda,\dots H_{\nu_s}\circ\Lambda)\Bigr]$$

depends only on Λ and H_γ with $\gamma < \nu$ because $2 \le s \le \nu - 1$ in the sum. In particular, if $H_1, \ldots, H_{\nu-1} \in \mathcal{H}_\Lambda$ then $Q_\nu = O$, and (i) is proved.

Now let us study $S_1(\nu)$. First of all, we clearly have $S_1(\nu) = O$ for $2 \le \nu < \mu$, and $S_1(\mu) = F_\mu$. When $\nu > \mu$ we can write

$$\begin{aligned} S_1(\nu) &= F_\nu + \sum_{\substack{\mu \le r \le \nu-1 \\ d_1+\cdots+d_r=\nu}} \tilde{F}_r(H_{d_1}, \ldots, H_{d_r}) \\ &= F_\nu + \mu \tilde{F}_\mu(H_{\nu-\mu+1}, I, \ldots, I) + \sum_{\substack{d_1+\cdots+d_\mu=\nu \\ 1<\max\{d_j\}<\nu-\mu+1}} \tilde{F}_\mu(H_{d_1}, \ldots, H_{d_\mu}) + \sum_{\substack{\mu+1 \le r \le \nu-1 \\ d_1+\cdots+d_r=\nu}} \tilde{F}_r(H_{d_1}, \ldots, H_{d_r}) \;. \end{aligned}$$

in particular, $S_1(\mu+1) = F_{\mu+1} + \mu \tilde{F}_\mu(H_2, I, \ldots, I)$. Notice that the two remaining sums depend only on F_ρ with $\rho < \nu$ and on H_γ with $\gamma < \nu - \mu + 1$ (in the first sum is clear; for the second one, if $d_j \ge \nu - \mu + 1$ for some j we then would have $d_1 + \cdots + d_r \ge \nu - \mu + 1 + r - 1 \ge \nu + 1$, impossible). Summing up we have

$$S_1(\nu) = \begin{cases} O & \text{for } 2 \le \nu < \mu, \\ F_\mu & \text{for } \nu = \mu, \\ F_{\mu+1} + \mu \tilde{F}_\mu(H_2, I, \ldots, I) & \text{for } \nu = \mu + 1, \\ F_\nu + \mu \tilde{F}_\mu(H_{\nu-\mu+1}, I, \ldots, I) + R^1_\nu & \text{for } \nu > \mu + 1, \end{cases}$$

where

$$R^1_\nu = \sum_{\substack{d_1+\cdots+d_\mu=\nu \\ 1<\max\{d_j\}<\nu-\mu+1}} \tilde{F}_\mu(H_{d_1}, \ldots, H_{d_\mu}) + \sum_{\substack{\mu+1 \le r \le \nu-1 \\ d_1+\cdots+d_r=\nu}} \tilde{F}_r(H_{d_1}, \ldots, H_{d_r})$$

depends only on F_ρ with $\rho < \nu$ and on H_γ with $\gamma < \nu - \mu + 1$.

Let us now discuss $S_s(\nu)$ for $s \ge 2$. First of all, the condition $\max\{r_1, \ldots, r_s\} \ge \mu$ implies

$$\mu + s - 1 \le r_1 + \cdots + r_s \le \nu_1 + \cdots + \nu_s = \nu \;,$$

that is $s \le \nu - \mu + 1$. In particular, $S_s(\nu) = O$ if $\nu \le \mu$ or if $s > \nu - \mu + 1$. Moreover, if we had $d_{ij} \ge \nu - \mu + 1$ for some $1 \le i \le s$ and $1 \le j \le r_s$ we would get

$$\nu = d_{11} + \cdots + d_{sr_s} \ge \nu - \mu + 1 + r_1 + \cdots + r_s - 1 \ge \nu - \mu + 1 + \mu + s - 1 - 1 = \nu + s - 1 > \nu \;,$$

impossible. This means that $S_s(\nu)$ depends only on F_ρ with $\rho < \nu$ for all s, on H_γ with $\gamma < \nu - \mu + 1$ when $s < \nu - \mu + 1$, and that $S_{\nu-\mu+1}(\nu)$ depends on $H_{\nu-\mu+1}$ only because it contains $\tilde{K}_{\nu-\mu+1}$. Furthermore, the conditions $\max\{r_1, \ldots, r_{\nu-\mu+1}\} \ge \mu$ and $\nu_1 + \ldots + \nu_{\nu-\mu+1} = \nu$ imply that

$$S_{\nu-\mu+1}(\nu) = (\nu - \mu + 1) \tilde{K}_{\nu-\mu+1}(F_\mu, \Lambda, \ldots, \Lambda) = -(\nu - \mu + 1) \tilde{H}_{\nu-\mu+1}(F_\mu, \Lambda, \ldots, \Lambda) + R^2_\nu \;,$$

where (using Lemmas 1.2 and 2.2)

$$R^2_\nu = \sum_{\substack{2 \le r \le \nu-\mu \\ d_1+\cdots+d_r=\nu-\mu+1}} \sum_{j=1}^r d_j \tilde{K}_r\big(H_{d_1} \circ \Lambda, \ldots, \tilde{H}_{d_j}(F_\mu, \Lambda, \ldots, \Lambda), \ldots, H_{d_r} \circ \Lambda\big)$$

depends only on Λ, F_μ and H_γ with $\gamma < \nu - \mu + 1$.

Putting everything together, we have

$$\begin{aligned}\{\Phi^{-1}\circ F\circ\Phi\}_\nu &= T_\nu + S_1(\nu) + \sum_{s=2}^{\nu-\mu+1} S_s(\nu)\\ &= F_\nu - L_\Lambda(H_\nu) + Q_\nu + \begin{cases} O & \text{if } 2\le\nu\le\mu,\\ -L_{F_\mu,\Lambda}(H_2) & \text{if } \nu=\mu+1,\\ -L_{F_\mu,\Lambda}(H_{\nu-\mu+1}) + R_\nu & \text{if } \nu>\mu+1,\end{cases}\end{aligned}$$

where

$$R_\nu = R_\nu^1 + R_\nu^2 + \sum_{s=2}^{\nu-\mu} S_s(\nu)$$

depends only on F_ρ with $\rho<\mu$ and on H_γ with $\gamma<\nu-\mu+1$. In particular, if $F=\Lambda$ then we have $S_s(\nu)=O$ for all $s\ge 1$ and hence $R_\nu=O$ for all $\nu\ge 2$.

In this way we have proved (2.4) and parts (i), (ii) and (iii). Concerning (iv), it suffices to notice that if $F_2,\dots,F_{\nu-1}$ and $H_2,\dots,H_{\nu-\mu+1}$ are Λ-resonant, then also R_ν^1, $S_2(\nu),\dots,S_{\nu-\mu}(\nu)$ and R_ν^2 (by Lemmas 1.3 and 2.2) are Λ-resonant. □

We can now prove the existence of a first order normalization in the sense described in the introduction.

Proposition 2.4: *Take $\Lambda\in M_{n,n}(\mathbb{C})$ and for each $\nu\ge 2$ choose two subspaces $\mathcal{N}^\nu$, $\mathcal{M}^\nu\subseteq\mathcal{H}^\nu$ such that $\mathcal{H}^\nu=\operatorname{Im}L_\Lambda|_{\mathcal{H}^\nu}\oplus\mathcal{N}^\nu$ and $\mathcal{H}^\nu=\mathcal{H}^\nu_\Lambda\oplus\mathcal{M}^\nu$. Then for every $F\in\widehat{\mathcal{O}}^n_\Lambda$ there exists a unique $\Phi=I+\sum\limits_{d\ge 2}H_d$ such that $H_d\in\mathcal{M}^d$ for all $d\ge 2$ and $\{\Phi^{-1}\circ F\circ\Phi\}_\nu\in\mathcal{N}^\nu$ for all $\nu\ge 2$.*

Proof: Notice that, by construction, $L_\Lambda(\mathcal{H}^\nu)=L_\Lambda(\mathcal{M}^\nu)$ and $L_\Lambda|_{\mathcal{M}^\nu}$ is injective. Now put $G=\Phi^{-1}\circ F\circ\Phi$; we define H_d by induction. For $d=2$, we see that there exist a unique $G\in\mathcal{N}^2$ and a unique $H\in\mathcal{M}^2$ such that $F_2=G+L_\Lambda(H)$. Since (2.4) says that

$$G_2 = F_2 - L_\Lambda(\{\Phi\}_2) = G + L_\Lambda(H) - L_\Lambda(\{\Phi\}_2)\ ,$$

to get $G_2\in\mathcal{N}^2$ with $\{\Phi\}_2\in\mathcal{M}^2$ we must take $\{\Phi\}_2=H$.

Assume now that we have defined $H_j\in\mathcal{M}^j$ for $j=2,\dots,H_{d-1}$. In particular, this determines completely the terms Q_d, R_d and $L_{F_2,\Lambda}(H_{d-1})$ in (2.4). So there exist a unique $G\in\mathcal{N}^d$ and a unique $H\in\mathcal{M}^d$ such that $F_d-L_{F_2,\Lambda}(H_{d-1})+Q_d+R_d=G+L_\Lambda(H)$. Then to get $G_d\in\mathcal{N}^d$ with $\{\Phi\}_d\in\mathcal{M}^d$ the only choice is $\{\Phi\}_d=H$, and thus $G_d=G$. □

There are a few natural choices for the subspaces $\mathcal{N}^\nu$ and $\mathcal{M}^\nu$ (see, e.g., [Mu1, Chapter 4]). If Λ is diagonal, then Remark 1.1 shows that we can take $\mathcal{N}^\nu=\mathcal{H}^\nu_\Lambda$ and $\mathcal{M}^\nu=\operatorname{Im}L_\Lambda|_{\mathcal{H}^\nu}$, and thus Proposition 2.4 gives nothing but the usual Poincaré-Dulac normal form.

Another possibility arises choosing on each $\mathcal{H}^\nu$ a positive definite Hermitian product. Then, denoting by L^*_Λ the adjoint operator of L_Λ, we have

$$\mathcal{H}^\nu = \operatorname{Im}L_\Lambda|_{\mathcal{H}^\nu}\bigoplus\operatorname{Ker}L^*_\Lambda|_{\mathcal{H}^\nu} = \mathcal{H}^\nu_\Lambda\bigoplus\operatorname{Im}L^*_\Lambda|_{\mathcal{H}^\nu}\ ,$$

and thus we can take $\mathcal{N}^\nu=\operatorname{Ker}L^*_\Lambda|_{\mathcal{H}^\nu}$ and $\mathcal{M}^\nu=\operatorname{Im}L^*_\Lambda|_{\mathcal{H}^\nu}$.

If we use the Fischer Hermitian product introduced in (0.3), it turns out that $L^*_\Lambda=L_{\Lambda^*}$, where Λ^* is the matrix adjoint of Λ (see, e.g., [Mu1, Lemma 4.6.6]). Furthermore, when Λ is diagonal we clearly have $\operatorname{Ker}L_{\Lambda^*}=\operatorname{Ker}L_\Lambda$, and thus we have again recovered the usual Poincaré-Dulac normal form. More generally, if $\Lambda=D+N$ is in Jordan normal form, with D diagonal and N nilpotent, then $\operatorname{Ker}L^*_\Lambda=\operatorname{Ker}L_D\cap\operatorname{Ker}L_{N^*}\subseteq\operatorname{Ker}L_D$ (see, e.g., [Mu1, Lemma 4.6.9]), and thus in this case too we have recovered the usual Poincaré-Dulac normal form (composed by monomials resonant with respect to the eigenvalues of Λ, i.e., D-resonant).

We can now introduce the renormalized Poincaré-Dulac normal form.

Definition 2.4: Given $\Lambda\in M_{n,n}(\mathbb{C})$, we shall say that a $G\in\widehat{\mathcal{O}}^n_\Lambda$ is *in renormalized Poincaré-Dulac normal form* if $G=\Lambda$ or the homogeneous expansion $G=\Lambda+\sum\limits_{d\ge\mu}G_d$ of G satisfies the following conditions:

(a) $G_\mu\in\mathcal{H}^\mu_\Lambda\setminus\{O\}$;
(b) $G_d\in\mathcal{H}^d_\Lambda\cap(\operatorname{Im}L_{G_\mu,\Lambda})^\perp$ for all $d>\mu$ (where we are using the Fischer Hermitian product).

Given $F\in\widehat{\mathcal{O}}^n_\Lambda$, we shall say that $G\in\widehat{\mathcal{O}}^n_\Lambda$ is *a renormalized Poincaré-Dulac normal form* of F if G is in renormalized Poincaré-Dulac normal form and $G=\Phi^{-1}\circ F\circ\Phi$ for some $\Phi\in\widehat{\mathcal{O}}^n_I$.

To proceed with the renormalization as explained in the introduction, we need condition (0.2), that is we need to check that the operator $L_{F_\mu,\Lambda}$ with $F_\mu\in\mathcal{N}^\mu$ sends $\mathcal{H}^{\nu-\mu+1}_\Lambda$ into $\mathcal{N}^\nu$. When Λ is diagonal, we have $\mathcal{N}^\nu=\mathcal{H}^\nu_\Lambda$ for all $\nu\ge2$, and hence (0.2) follows from Corollary 1.4. We then have the renormalized normal form we were looking for:

Theorem 2.5: *Let $\Lambda\in M_{n,n}(\mathbb{C})$ be diagonal. Then each $F\in\widehat{\mathcal{O}}^n_\Lambda$ admits a renormalized Poincaré-Dulac normal form. More precisely, if $F=\Lambda+\sum\limits_{d\ge\mu}F_d$ is in Poincaré-Dulac normal form (and $F\not\equiv\Lambda$) then there exists a unique Λ-resonant $\Phi=I+\sum\limits_{d\ge2}H_d\in\widehat{\mathcal{O}}^n_I$ such that $H_d\in(\operatorname{Ker}L_{F_\mu,\Lambda})^\perp$ for all $d\ge2$ and $G=\Phi^{-1}\circ F\circ\Phi$ is in renormalized Poincaré-Dulac normal form.*

Proof: By Proposition 2.4 we can assume that F is in Poincaré-Dulac normal form. If $F\equiv\Lambda$ we are done; assume then that $F\not\equiv\Lambda$.

First of all, by Proposition 2.3 if Φ is Λ-resonant we have $\{\Phi^{-1}\circ F\circ\Phi\}_d=F_d$ for all $d\le\mu$. In particular, $F_\mu\in\mathcal{H}^\mu_\Lambda$; therefore, by Corollary 1.4, $\operatorname{Im}L_{F_\mu,\Lambda}|_{\mathcal{H}^{\nu-\mu+1}_\Lambda}\subseteq\mathcal{H}^\nu_\Lambda$. We then have the splittings

$$\mathcal{H}^\nu_\Lambda=\operatorname{Im}L_{F_\mu,\Lambda}|_{\mathcal{H}^{\nu-\mu+1}_\Lambda}\bigoplus(\operatorname{Im}L_{F_\mu,\Lambda}|_{\mathcal{H}^{\nu-\mu+1}_\Lambda})^\perp$$

and

$$\mathcal{H}^{\nu-\mu+1}_\Lambda=\operatorname{Ker}L_{F_\mu,\Lambda}|_{\mathcal{H}^{\nu-\mu+1}_\Lambda}\bigoplus(\operatorname{Ker}L_{F_\mu,\Lambda}|_{\mathcal{H}^{\nu-\mu+1}_\Lambda})^\perp\ .$$

Hence we can find a unique $G\in(\operatorname{Im}L_{F_\mu,\Lambda})^\perp\cap\mathcal{H}^{\mu+1}_\Lambda$ and a unique $H\in(\operatorname{Ker}L_{F_\mu,\Lambda})^\perp\cap\mathcal{H}^2_\Lambda$ such that $F_{\mu+1}=G+L_{F_\mu,\Lambda}(H)$. Then Proposition 2.3 yields

$$\{\Phi^{-1}\circ F\circ\Phi\}_{\mu+1}=F_{\mu+1}-L_{F_\mu,\Lambda}(\{\Phi\}_2)=G+L_{F_\mu,\Lambda}(H)-L_{F_\mu,\Lambda}(\{\Phi\}_2)\ ;$$

so to get $\{\Phi^{-1}\circ F\circ\Phi\}_{\mu+1}\in(\operatorname{Im}L_{F_\mu,\Lambda})^\perp\cap\mathcal{H}^{\mu+1}_\Lambda$ with $\{\Phi\}_2\in(\operatorname{Ker}L_{F_\mu,\Lambda})^\perp\cap\mathcal{H}^2_\Lambda$ we must necessarily take $\{\Phi\}_2=H$.

Assume, by induction, that we have uniquely determined $H_2,\dots,H_{d-1}\in(\operatorname{Im}L_{F_\mu,\Lambda})^\perp\cap\mathcal{H}_\Lambda$; in particular, this determines completely $R_d\in\mathcal{H}^d_\Lambda$ in (2.4). Hence there exist a unique G in $(\operatorname{Im}L_{F_\mu,\Lambda})^\perp\cap\mathcal{H}^d_\Lambda$ and a unique $H\in(\operatorname{Ker}L_{F_\mu,\Lambda})^\perp\cap\mathcal{H}^{d-\mu+1}_\Lambda$ such that $F_d+R_d=G+L_{F_\mu,\Lambda}(H)$. So to get $\{\Phi^{-1}\circ F\circ\Phi\}_d\in(\operatorname{Im}L_{F_\mu,\Lambda})^\perp\cap\mathcal{H}^d_\Lambda$ with $\{\Phi\}_{d-\mu+1}\in(\operatorname{Ker}L_{F_\mu,\Lambda})^\perp\cap\mathcal{H}^{d-\mu+1}_\Lambda$ the only choice is $\{\Phi\}_{d-\mu+1}=H$, and thus $\{\Phi^{-1}\circ F\circ\Phi\}_d=G$. □

As examples of applications of this method, in the remaining two sections we shall study cases where the usual Poincaré-Dulac normal form reduces to the original map.

3. Examples with $\Lambda=O$

In this section we shall completely describe the renormalized normal forms obtained when $n=\mu=2$ and $\Lambda=O$, that is in the 2-dimensional quadratic superattracting case. It is worthwhile to remark

that, except in a few degenerate instances, the normal form will be expressed just in terms of two power series of *one* variable, and thus we shall obtain a drastic simplification of the germs.

In [A3] we showed that, up to a linear change of variable, we can assume that the quadratic term F_2 is of one (and only one) of the following forms:

(∞) $F_2(z,w)=(z^2,zw)$;
(1_{00}) $F_2(z,w)=(0,-z^2)$;
(1_{10}) $F_2(z,w)=\bigl(-z^2,-(z^2+zw)\bigr)$;
(1_{11}) $F_2(z,w)=\bigl(-zw,-(z^2+w^2)\bigr)$;
(2_{001}) $F_2(z,w)=(0,zw)$;
(2_{011}) $F_2(z,w)=(zw,zw+w^2)$;
$(2_{10\rho})$ $F_2(z,w)=\bigl(-\rho z^2,(1-\rho)zw\bigr)$, with $\rho\neq 0$;
$(2_{11\rho})$ $F_2(z,w)=\bigl(\rho z^2+zw,(1+\rho)zw+w^2\bigr)$, with $\rho\neq 0$;
(3_{100}) $F_2(z,w)=(z^2-zw,0)$;
$(3_{\rho 10})$ $F_2(z,w)=\bigl(\rho(-z^2+zw),(1-\rho)(zw-w^2)\bigr)$, with $\rho\neq 0,\,1$;
$(3_{\rho\tau 1})$ $F_2(z,w)=\bigl(-\rho z^2+(1-\tau)zw,(1-\rho)zw-\tau w^2\bigr)$, with $\rho,\,\tau\neq 0$ and $\rho+\tau\neq 1$

(where the symbols refer to the number of characteristic directions and to their indeces; see also [AT2]).

We shall use the standard basis $\{u_{d,j},v_{d,j}\}_{j=0,\ldots,d}$ of $\mathcal{H}^d$, where

$$u_{d,j}=(z^jw^{d-j},0)\qquad\text{and}\qquad v_{d,j}=(0,z^jw^{d-j})\;,$$

and we shall endow $\mathcal{H}^d$ with the usual Fischer scalar product, so that $\{u_{d,j},v_{d,j}\}_{j=0,\ldots,d}$ is an orthogonal basis and

$$\|u_{d,j}\|^2=\|v_{d,j}\|^2=\binom{d}{j}^{-1}\;.$$

Finally, we recall that when $\Lambda=O$ the operator $L=L_{F_2,\Lambda}$ is given by

$$L(H)=-\operatorname{Jac}(F_2)\cdot H\;.$$

We shall now study separately each case.

• *Case* (∞).
In this case we have

$$L(u_{d,j})=-2u_{d+1,j+1}-v_{d+1,j}\qquad\text{and}\qquad L(v_{d,j})=-v_{d+1,j+1}$$

for all $d\geq 2$ and $j=0,\ldots,d$. Therefore

$$\operatorname{Im}L|_{\mathcal{H}^d}=\operatorname{Span}\left(u_{d+1,2},\ldots,u_{d+1,d+1},2u_{d+1,1}+v_{d+1,0},v_{d+1,1},\ldots,v_{d+1,d+1}\right)\;,$$

and thus

$$(\operatorname{Im}L|_{\mathcal{H}^d})^\perp=\operatorname{Span}\left(u_{d+1,0},(d+1)u_{d+1,1}-2v_{d+1,0}\right)\;.$$

It then follows that every formal power series of the form

$$F(z,w)=(z^2+O_3,zw+O_3)$$

(where O_3 denotes a remainder term of order at least 3) is formally conjugated to a power series of the form

$$G(z,w)=\bigl(z^2+\varphi(w)+z\psi'(w),zw-2\psi(w)\bigr)$$

where φ, $\psi\in\mathbb{C}[\![\zeta]\!]$ are arbitrary power series of order at least 3. Notice that (here and in later formulas) the appearance of the derivative (which simplifies the expression of the normal form) is due to the fact we are using the Fischer Hermitian product; using another Hermitian product might lead to more complicated normal forms.

• *Case* (1_{00}).
In this case we have

$$L(u_{d,j}) = 2v_{d+1,j+1} \qquad \text{and} \qquad L(v_{d,j}) = 0$$

for all $d \geq 2$ and $j = 0, \dots, d$. Therefore

$$\operatorname{Im} L|_{\mathcal{H}^d} = \operatorname{Span}\left(v_{d+1,1}, \dots, v_{d+1,d+1}\right) ,$$

and thus

$$(\operatorname{Im} L|_{\mathcal{H}^d})^{\perp} = \operatorname{Span}\left(u_{d+1,0}, \dots, u_{d+1,d+1}, v_{d+1,0}\right) .$$

It then follows that every formal power series of the form

$$F(z,w) = (O_3, -z^2 + O_3)$$

is formally conjugated to a power series of the form

$$G(z,w) = \big(\Phi(z,w), -z^2 + \psi(w)\big)$$

where $\psi \in \mathbb{C}[\![\zeta]\!]$ and $\Phi \in \mathbb{C}[\![z,w]\!]$ are arbitrary power series of order at least 3.

• *Case* (1_{10}).
In this case we have

$$L(u_{d,j}) = 2u_{d+1,j+1} + 2v_{d+1,j+1} + v_{d+1,j} \qquad \text{and} \qquad L(v_{d,j}) = v_{d+1,j+1}$$

for all $d \geq 2$ and $j = 0, \dots, d$. Therefore

$$\operatorname{Im} L|_{\mathcal{H}^d} = \operatorname{Span}\left(2u_{d+1,1} + v_{d+1,0}, u_{d+1,2}, \dots, u_{d+1,d+1}, v_{d+1,1}, \dots, v_{d+1,d+1}\right) ,$$

and thus

$$(\operatorname{Im} L|_{\mathcal{H}^d})^{\perp} = \operatorname{Span}\left(u_{d+1,0}, (d+1)u_{d+1,1} - 2v_{d+1,0}\right) .$$

It then follows that every formal power series of the form

$$F(z,w) = (-z^2 + O_3, -z^2 - zw + O_3)$$

is formally conjugated to a power series of the form

$$G(z,w) = \big(-z^2 + \varphi(w) + z\psi'(w), -z^2 - zw - 2\psi(w)\big)$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are arbitrary power series of order at least 3.

• *Case* (1_{11}).
In this case we have

$$L(u_{d,j}) = u_{d+1,j} + 2v_{d+1,j+1} \qquad \text{and} \qquad L(v_{d,j}) = u_{d+1,j+1} + 2v_{d+1,j}$$

for all $d \geq 2$ and $j = 0, \dots, d$. It follows that

$$\begin{aligned}\operatorname{Im} L|_{\mathcal{H}^d} = \operatorname{Span}(&u_{d+1,0} - u_{d+1,2}, \dots, u_{d+1,d-1} - u_{d+1,d+1},\\ &v_{d+1,2} - v_{d+1,0}, \dots, v_{d+1,d+1} - v_{d+1,d-1}, u_{d+1,0} + 2v_{d+1,1}, u_{d+1,1} + 2v_{d+1,0}) ,\end{aligned}$$

and a few computations yield

$$\begin{aligned}(\operatorname{Im} L|_{\mathcal{H}^d})^\perp &= \operatorname{Span}\left(\sum_{j=0}^{d+1}\binom{d+1}{j}(v_{d+1,j}-2u_{d+1,j}),\sum_{j=0}^{d+1}(-1)^j\binom{d+1}{j}(v_{d+1,j}+2u_{d+1,j})\right)\;.\\ &= \operatorname{Span}\big(\big(-2(z+w)^{d+1},(z+w)^{d+1}\big),\big(2(w-z)^{d+1},(w-z)^{d+1}\big)\big)\end{aligned}$$

It then follows that every formal germ of the form

$$F(z,w)=(-zw+O_3,-z^2-w^2+O_3)$$

is formally conjugated to a germ of the form

$$G(z,w)=\big(-zw-2\varphi(z+w)+2\psi(w-z),-z^2-w^2+\varphi(z+w)+\psi(w-z)\big)$$

where φ, $\psi\in\mathbb{C}[\![\zeta]\!]$ are arbitrary power series of order at least 3. Again, the fact that the normal form is expressed in terms of power series evaluated in $z+w$ and $z-w$ is due to the fact we are using the Fischer Hermitian product.

• *Case* (2_{001}).
In this case we have

$$L(u_{d,j})=-v_{d+1,j}\qquad\text{and}\qquad L(v_{d,j})=-v_{d+1,j+1}$$

for all $d\geq 2$ and $j=0,\ldots,d$. It follows that

$$\operatorname{Im} L|_{\mathcal{H}^d}=\operatorname{Span}(v_{d+1,0},\ldots,v_{d+1,d+1})$$

and hence

$$(\operatorname{Im} L|_{\mathcal{H}^d})^\perp=\operatorname{Span}(u_{d+1,0},\ldots,u_{d+1,d+1})\;.$$

It then follows that every formal germ of the form

$$F(z,w)=(O_3,zw+O_3)$$

is formally conjugated to a germ of the form

$$G(z,w)=\big(\Phi(z,w),zw\big)$$

where $\Phi\in\mathbb{C}[\![z,w]\!]$ is a power series of order at least three.

• *Case* (2_{011}).
In this case we have

$$L(u_{d,j})=-u_{d+1,j}-v_{d+1,j}\qquad\text{and}\qquad L(v_{d,j})=-u_{d+1,j+1}-2v_{d+1,j}-v_{d+1,j+1}$$

for all $d\geq 2$ and $j=0,\ldots,d$. It follows that

$$\begin{aligned}\operatorname{Im} L|_{\mathcal{H}^d}=\operatorname{Span}(&u_{d+1,0},\ldots,u_{d+1,d-1},v_{d+1,0},\ldots,v_{d+1,d-1},\\ &u_{d+1,d}+v_{d+1,d},u_{d+1,d+1}+v_{d+1,d+1}+2v_{d+1,d})\;,\end{aligned}$$

and hence

$$(\operatorname{Im} L|_{\mathcal{H}^d})^\perp=\operatorname{Span}((d+1)u_{d+1,d}-(d+1)v_{d+1,d}+2v_{d+1,d+1},u_{d+1,d+1}-v_{d+1,d+1})\;.$$

It then follows that every formal germ of the form

$$F(z,w)=(zw+O_3,zw+w^2+O_3)$$

is formally conjugated to a germ of the form

$$G(z,w)=\big(zw+w\varphi'(z)+\psi(z),zw+w^2+2\varphi(z)-w\varphi'(z)-\psi(z)\big)\;,$$

where φ, $\psi\in\mathbb{C}[\![\zeta]\!]$ are arbitrary power series of order at least 3.

• *Case* $(2_{10\rho})$.
In this case we have

$$L(u_{d,j}) = 2\rho u_{d+1,j+1} + (\rho-1)v_{d+1,j} \qquad \text{and} \qquad L(v_{d,j}) = (\rho-1)v_{d+1,j+1}$$

for all $d \geq 2$ and $j = 0, \ldots, d$. We clearly have two subcases to consider.

If $\rho = 1$ then

$$\operatorname{Im} L|_{\mathcal{H}^d} = \operatorname{Span}\left(u_{d+1,1}, \ldots, u_{d+1,d+1}\right) ,$$

and hence

$$(\operatorname{Im} L|_{\mathcal{H}^d})^\perp = \operatorname{Span}\left(u_{d+1,0}, v_{d+1,0}, \ldots, v_{d+1,d+1}\right) .$$

It then follows that every formal germ of the form

$$F(z,w) = (-z^2 + O_3, O_3)$$

is formally conjugated to a germ of the form

$$G(z,w) = \left(-z^2 + \psi(w), \Phi(z,w)\right) ,$$

where $\psi \in \mathbb{C}[\![\zeta]\!]$ and $\Phi \in \mathbb{C}[\![z,w]\!]$ are arbitrary power series of order at least 3.

If instead $\rho \neq 1$ (recalling that $\rho \neq 0$ too) then

$$\operatorname{Im} L|_{\mathcal{H}^d} = \operatorname{Span}\left(2\rho u_{d+1,1} + (\rho-1)v_{d+1,0}, u_{d+1,2}, \ldots, u_{d+1,d+1}, v_{d+1,1}, \ldots, v_{d+1,d+1}\right) ,$$

and hence

$$(\operatorname{Im} L|_{\mathcal{H}^d})^\perp = \operatorname{Span}\left(u_{d+1,0}, (\rho-1)(d+1)u_{d+1,1} - 2\rho v_{d+1,0}\right) .$$

It then follows that every formal germ of the form

$$F(z,w) = \left(-\rho z^2 + O_3, (1-\rho)zw + O_3\right)$$

with $\rho \neq 0$, 1 is formally conjugated to a germ of the form

$$G(z,w) = \left(-\rho z^2 + (\rho-1)z\varphi'(w) + \psi(w), (1-\rho)zw - 2\rho\varphi(z)\right) ,$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are arbitrary power series of order at least 3.

• *Case* $(2_{11\rho})$.
In this case we have

$$\begin{cases} L(u_{d,j}) = -2\rho u_{d+1,j+1} - u_{d+1,j} - (1+\rho)v_{d+1,j} \\ L(v_{d,j}) = -u_{d+1,j+1} - 2v_{d+1,j} - (1+\rho)v_{d+1,j+1} \end{cases} \tag{3.1}$$

for all $d \geq 2$ and $j = 0, \ldots, d$. We clearly have two subcases to consider.

If $\rho = -1$ then

$$\operatorname{Im} L|_{\mathcal{H}^d} = \operatorname{Span}\left(u_{d+1,0} - 2u_{d+1,1}, \ldots, u_{d+1,d} - 2u_{d+1,d+1}, u_{d+1,1} + 2v_{d+1,0}, \ldots, u_{d+1,d} + 2v_{d+1,d}\right) ,$$

and hence

$$\begin{aligned}(\operatorname{Im} L|_{\mathcal{H}^d})^\perp &= \operatorname{Span}\left(\sum_{j=0}^{d+1} \binom{d+1}{j} \frac{1}{2^j}(u_{d+1,j} - \frac{1}{4}v_{d+1,j}), v_{d+1,d+1}\right) \\ &= \operatorname{Span}\left(\left(\left(\tfrac{z}{2}+w\right)^{d+1}, -\frac{1}{4}\left(\tfrac{z}{2}+w\right)^{d+1}\right), (0, z^{d+1})\right) .\end{aligned}$$

It then follows that every formal germ of the form

$$F(z,w) = (-z^2 + zw + O_3, w^2 + O_3)$$

is formally conjugated to a germ of the form

$$G(z,w) = \left(-z^2 + zw + \varphi(\tfrac{z}{2} + w), w^2 - \frac{1}{4}\varphi(\tfrac{z}{2} + w) + \psi(z)\right) ,$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are arbitrary power series of order at least 3.

If instead $\rho \neq -1$ (recalling that $\rho \neq 0$ too) then a basis of $\operatorname{Im} L|_{\mathcal{H}^d}$ is given by the vectors listed in (3.1), and a computation shows that $(\operatorname{Im} L|_{\mathcal{H}^d})^\perp$ is given by homogeneous maps of the form

$$\sum_{j=0}^{d+1}(a_j u_{d+1,j} + b_j v_{d+1,j})$$

where the coefficients a_j, b_j satisfy the following relations:

$$\begin{cases} c_j b_j = -\dfrac{2}{1+\rho} c_{j-1} b_{j-1} - \dfrac{1}{\rho(1+\rho)} c_{j-2} b_{j-2} & \text{for } j = 2, \dots, d+1, \\ c_j a_j = \dfrac{1}{\rho} c_{j-2} b_{j-2} & \text{for } j = 2, \dots, d+1, \\ a_0 = (3\rho - 1) b_0 + 2\dfrac{\rho(1+\rho)}{d+1} b_1 \ , \\ a_1 = -2(d+1) b_0 - (1+\rho) b_1 \ , \end{cases}$$

where $c_j^{-1} = \binom{d+1}{j}$ and b_0, $b_1 \in \mathbb{C}$ are arbitrary. Solving this recurrence equation one gets

$$b_j = \frac{1}{2\sqrt{-\rho}} \binom{d+1}{j} \left[\frac{\rho(1+\rho)}{d+1} (m_\rho^j - n_\rho^j) b_1 + \big(\rho(m_\rho^j - n_\rho^j) + \sqrt{-\rho}(m_\rho^j + n_\rho^j)\big) b_0 \right] ,$$

where $\sqrt{-\rho}$ is any square root of $-\rho$, and

$$m_\rho = \frac{\sqrt{-\rho} - \rho}{\rho(1+\rho)} \ , \qquad n_\rho = -\frac{\sqrt{-\rho} + \rho}{\rho(1+\rho)} \ .$$

It follows that the renormalized normal form of a formal germ of the form

$$F(z,w) = \big(\rho z^2 + zw + O_3, (1+\rho) zw + w^2 + O_3\big)$$

with $\rho \neq 0$, -1 is

$$\begin{aligned} G(z,w) = \Bigg(& \rho z^2 + zw + \frac{1}{\rho}\left[\frac{1 - \sqrt{-\rho}}{2 m_\rho^2} \varphi(m_\rho z + w) + \frac{1 + \sqrt{-\rho}}{2 n_\rho^2} \varphi(n_\rho z + w) \right] \\ & \qquad + \frac{1+\rho}{2\sqrt{-\rho}} \left(\frac{1}{m_\rho^2} \psi(m_\rho z + w) - \frac{1}{n_\rho^2} \psi(n_\rho z + w) \right), \\ & (1+\rho) zw + w^2 + \frac{1 - \sqrt{-\rho}}{2} \varphi(m_\rho z + w) + \frac{1 + \sqrt{-\rho}}{2} \varphi(n_\rho z + w) \\ & \qquad + \frac{\rho(1+\rho)}{2\sqrt{-\rho}} \big(\psi(m_\rho z + w) - \psi(n_\rho z + w) \big) \Bigg) \end{aligned}$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3.

• *Case* (3_{100}).
In this case we have

$$L(u_{d,j})=u_{d+1,j}-2u_{d+1,j+1}\qquad\text{and}\qquad L(v_{d,j})=u_{d+1,j+1}$$

for all $d\ge 2$ and $j=0,\dots,d$. It follows that

$$\operatorname{Im}L|_{\mathcal{H}^d}=\operatorname{Span}(u_{d+1,0},\dots,u_{d+1,d+1})$$

and hence

$$(\operatorname{Im}L|_{\mathcal{H}^d})^\perp=\operatorname{Span}(v_{d+1,0},\dots,v_{d+1,d+1})\ .$$

It then follows that every formal germ of the form

$$F(z,w)=(z^2-zw+O_3,O_3)$$

is formally conjugated to a germ of the form

$$G(z,w)=\bigl(z^2-zw,\Phi(z,w)\bigr)\ ,$$

where $\Phi\in\mathbb{C}[\![z,w]\!]$ is a power series of order at least 3.

• *Case* $(3_{\rho 10})$.
In this case we have

$$\begin{cases}L(u_{d,j})=\rho(2u_{d+1,j+1}-u_{d+1,j})+(\rho-1)v_{d+1,j}\\ L(v_{d,j})=-\rho u_{d+1,j+1}+(\rho-1)(v_{d+1,j+1}-2v_{d+1,j})\end{cases}\tag{3.2}$$

for all $d\ge 2$ and $j=0,\dots,d$. Then a basis of $\operatorname{Im}L|_{\mathcal{H}^d}$ is given by the homogeneous maps listed in (3.2), and a computation shows that $(\operatorname{Im}L|_{\mathcal{H}^d})^\perp$ is given by homogeneous maps of the form

$$\sum_{j=0}^{d+1}(a_ju_{d+1,j}+b_jv_{d+1,j})$$

where the coefficients a_j, b_j satisfy the following relations:

$$\begin{cases}c_{j+1}a_{j+1}=\dfrac{\rho-1}{\rho}(c_{j+1}b_{j+1}-2c_jb_j) & \text{for } j=0,\dots,d,\\ c_{j+1}b_{j+1}=2c_jb_j-c_{j-1}b_{j-1} & \text{for } j=1,\dots,d,\\ c_0a_0=2c_1a_1+\dfrac{\rho-1}{\rho}c_0b_0\ , & \end{cases}$$

where $c_j^{-1}=\binom{d+1}{j}$ and b_0, $b_1\in\mathbb{C}$ are arbitrary. Solving this recurrence equation we find

$$\begin{cases}b_j=\binom{d+1}{j}\left[\frac{j}{d+1}b_1-(j-1)b_0\right] & \text{for } j=0,\dots,d+1,\\ a_j=\frac{\rho-1}{\rho}\binom{d+1}{j}\left[\frac{2-j}{d+1}b_1+(j-3)b_0\right] & \text{for } j=0,\dots,d+1,\end{cases}$$

where b_0, $b_1\in\mathbb{C}$ are arbitrary. So every formal germ of the form

$$F(z,w)=\bigl(\rho(-z^2+zw)+O_3,(1-\rho)(zw-w^2)+O_3\bigr)$$

with $\rho\ne 0$, 1 is formally conjugated to a germ of the form

$$\begin{aligned}G(z,w)=\Biggl(&\rho(-z^2+zw)+z\frac{\partial}{\partial z}\bigl[\varphi(z+w)+\psi(z+w)\bigr]-\varphi(z+w),\\ &(1-\rho)(zw-w^2)+\frac{\rho-1}{\rho}\left(z\frac{\partial}{\partial z}\bigl[\varphi(z+w)-\psi(z+w)\bigr]-3\varphi(z+w)+2\psi(z+w)\right)\Biggr)\end{aligned}$$

where φ, $\psi\in\mathbb{C}[\![\zeta]\!]$ are power series of order at least 3.

• *Case* $(3_{\rho\tau 1})$.
In this case we have

$$L(u_{d,j}) = (\tau-1)u_{d+1,j} + 2\rho u_{d+1,j+1} + (\rho-1)v_{d+1,j}$$

and

$$L(v_{d,j}) = (\tau-1)u_{d+1,j+1} + 2\tau v_{d+1,j} + (\rho-1)v_{d+1,j+1}$$

for all $d \geq 2$ and $j = 0, \dots, d$. As before, we have a few subcases to consider.

Assume first $\rho = \tau = 1$. Then

$$\operatorname{Im} L|_{\mathcal{H}^d} = \operatorname{Span}\left(u_{d+1,1}, \dots, u_{d+1,d+1}, v_{d+1,0}, \dots, v_{d+1,d}\right) \; ;$$

hence

$$(\operatorname{Im} L|_{\mathcal{H}^d})^\perp = \operatorname{Span}\left(u_{d+1,0}, v_{d+1,d+1}\right) \; ,$$

It then follows that every formal germ of the form

$$F(z,w) = (-z^2 + O_3, -w^2 + O_3)$$

is formally conjugated to a germ of the form

$$G(z,w) = \bigl(-z^2 + \varphi(w), -w^2 + \psi(z)\bigr) \; ,$$

where φ, $\psi \in \mathbb{C}[\![z,w]\!]$ are arbitrary power series of order at least 3.

Assume now $\rho \neq 1$. Then a computation shows that $(\operatorname{Im} L|_{\mathcal{H}^d})^\perp$ is given by homogeneous maps of the form

$$\sum_{j=0}^{d+1} (a_j u_{d+1,j} + b_j v_{d+1,j})$$

where the coefficients a_j, b_j satisfy the following relations:

$$\begin{cases} c_{j+1}a_{j+1} = \dfrac{\tau}{\rho} c_{j-1}b_{j-1} & \text{for } j = 1, \dots, d, \\ c_{j+1}b_{j+1} = -\dfrac{2\tau}{\rho-1} c_j b_j - \dfrac{\tau(\tau-1)}{\rho(\rho-1)} c_{j-1}b_{j-1} & \text{for } j = 1, \dots, d, \\ (\tau-1)c_1a_1 + (\rho-1)c_1b_1 + 2\tau c_0b_0 = 0 \; , \\ (\tau-1)c_0a_0 + (\rho-1)c_0b_0 + 2\rho c_1a_1 = 0 \; , \end{cases} \tag{3.3}$$

where $c_j^{-1} = \binom{d+1}{j}$ and b_0, $b_1 \in \mathbb{C}$ are arbitrary.

When $\tau = 1$ conditions (3.3) reduce to

$$\begin{cases} c_{j+1}a_{j+1} = \dfrac{1}{\rho} c_{j-1}b_{j-1} & \text{for } j = 1, \dots, d, \\ c_{j+1}b_{j+1} = -\dfrac{2}{\rho-1} c_j b_j & \text{for } j = 1, \dots, d, \\ (\rho-1)c_1b_1 + 2c_0b_0 = 0 \; , \\ (\rho-1)c_0b_0 + 2\rho c_1a_1 = 0 \; , \end{cases}$$

whose solution is

$$\begin{cases} a_j = \binom{d+1}{j}\dfrac{1}{\rho}\left(\dfrac{2}{1-\rho}\right)^{j-2} b_0 & \text{for } j=1,\dots,d+1,\\ b_j = \binom{d+1}{j}\left(\dfrac{2}{1-\rho}\right)^{j} b_0 & \text{for } j=0,\dots,d+1, \end{cases}$$

where a_0, $b_0 \in \mathbb{C}$ are arbitrary. Therefore

$$(\operatorname{Im} L|_{\mathcal{H}^d})^\perp = \operatorname{Span}\left((w^{d+1},0), \left(\frac{(1-\rho)^2}{4\rho}\left(\frac{2}{1-\rho}z+w\right)^{d+1}, \left(\frac{2}{1-\rho}z+w\right)^{d+1}\right)\right) ,$$

and thus every formal germ of the form

$$F(z,w) = \bigl(-\rho z^2 + O_3, (1-\rho)zw - w^2 + O_3\bigr)$$

with $\rho \neq 1$ is formally conjugated to a germ of the form

$$G(z,w) = \left(-\rho z^2 + \varphi(w) + \frac{(1-\rho)^2}{4\rho}\psi\left(\frac{2}{1-\rho}z+w\right), (1-\rho)zw - w^2 + \psi\left(\frac{2}{1-\rho}z+w\right)\right) ,$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are arbitrary power series of order at least 3.

The case $\rho = 1$ and $\tau \neq 1$ is treated in the same way; we get that every formal germ of the form

$$F(z,w) = \bigl(-z^2 + (1-\tau)zw + O_3, -\tau w^2 + O_3\bigr)$$

with $\tau \neq 1$ is formally conjugated to a germ of the form

$$G(z,w) = \left(-z^2 + (1-\tau)zw + \psi\left(\frac{1-\tau}{2}z+w\right), -\tau w^2 + \varphi(z) + \frac{(1-\tau)^2}{4\tau}\psi\left(\frac{1-\tau}{2}z+w\right)\right) ,$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are arbitrary power series of order at least 3.

Finally assume ρ, $\tau \neq 1$. Solving the recurrence equation (3.3) we find

$$\begin{aligned} b_j = \frac{1}{2\sqrt{\rho\tau(\rho+\tau-1)}}\binom{d+1}{j}\Biggl[&\frac{\rho(\rho-1)}{d+1}(m_{\rho,\tau}^j - n_{\rho,\tau}^j)b_1 \\ &+ \bigl(\rho\tau(m_{\rho,\tau}^j - n_{\rho,\tau}^j) + \sqrt{\rho\tau(\rho+\tau-1)}(m_{\rho,\tau}^j + n_{\rho,\tau}^j)\bigr)b_0\Biggr] , \end{aligned}$$

for $j = 0,\dots,d+1$, where $\sqrt{\rho\tau(\rho+\tau-1)}$ is any square root of $\rho\tau(\rho+\tau-1)$, and

$$m_{\rho,\tau} = \frac{\sqrt{\rho\tau(\rho+\tau-1)} - \rho\tau}{\rho(\rho-1)} , \qquad n_{\rho,\tau} = -\frac{\sqrt{\rho\tau(\rho+\tau-1)} + \rho\tau}{\rho(\rho-1)} .$$

Moreover, from (3.3) we also get

$$\begin{aligned} a_j = \frac{\tau}{2\rho\sqrt{\rho\tau(\rho+\tau-1)}}\binom{d+1}{j}\Biggl[&\frac{\rho(\rho-1)}{d+1}(m_{\rho,\tau}^{j-2} - n_{\rho,\tau}^{j-2})b_1 \\ &+ \bigl(\rho\tau(m_{\rho,\tau}^{j-2} - n_{\rho,\tau}^{j-2}) + \sqrt{\rho\tau(\rho+\tau-1)}(m_{\rho,\tau}^{j-2} + n_{\rho,\tau}^{j-2})\bigr)b_0\Biggr] , \end{aligned}$$

again for $j = 0, \ldots, d+1$. It follows that the renormalized normal form of a formal germ of the form

$$F(z,w) = \bigl(-\rho z^2 + (1-\tau)zw + O_3, (1-\rho)zw - \tau w^2 + O_3\bigr)$$

with ρ, $\tau \neq 0$, 1 and $\rho + \tau \neq 1$, is

$$\begin{aligned} G(z,w) = \Biggl(&-\rho z^2 + (1-\tau)zw + \frac{\tau}{\rho}\Biggl[\frac{\sqrt{\rho+\tau-1}+\sqrt{\rho\tau}}{2m_{\rho,\tau}^2}\varphi(m_{\rho,\tau}z+w) \\ &\qquad + \frac{\sqrt{\rho+\tau-1}-\sqrt{\rho\tau}}{2n_{\rho,\tau}^2}\varphi(n_{\rho,\tau}z+w) \\ &\qquad + \frac{1}{m_{\rho,\tau}^2}\psi(m_{\rho,\tau}z+w) - \frac{1}{n_{\rho,\tau}^2}\psi(n_{\rho,\tau}z+w)\Biggr], \\ &(1-\rho)zw - \tau w^2 + \frac{\sqrt{\rho+\tau-1}+\sqrt{\rho\tau}}{2}\varphi(m_{\rho,\tau}z+w) \\ &\qquad + \frac{\sqrt{\rho+\tau-1}-\sqrt{\rho\tau}}{2}\varphi(n_{\rho,\tau}z+w) \\ &\qquad + \psi(m_{\rho,\tau}z+w) - \psi(n_{\rho,\tau}z+w)\Biggr), \end{aligned}$$

where the square roots of $\rho\tau$ and of $\rho + \tau - 1$ are chosen so that their product is equal to the previously chosen square root of $\rho\tau(\rho+\tau-1)$, and φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3.

4. Examples with $\Lambda = I$

In this section we shall assume $n = \mu = 2$ and $\Lambda = I$, that is we shall be interested in 2-dimensional germs tangent to the identity of order 2. We shall keep using the notations introduced in the previous section. It should be recall that in his monumental work [É1] (see [É2] for a survey) Écalle studied the formal classification of germs tangent to the identity in dimension n, giving a complete set of formal invariants for germs satisfying a generic condition: the existence of at least one non-degenerate characteristic direction (an eigenradius, in Écalle's terminology). A *characteristic direction* of a germ tangent to the identity F is a non-zero direction v such that $F_\mu(v) = \lambda v$ for some $\lambda \in \mathbb{C}$, where F_μ is the first (nonlinear) non-vanishing term in the homogeneous expansion of F. The characteristic direction v is *degenerate* if $\lambda = 0$.

For this reason, we decided to discuss here the cases *without* non-degenerate characteristic directions, that is the cases (1_{00}), (1_{10}) and (2_{001}), that cannot be dealt with Écalle's methods. Furthermore, we shall also study the somewhat special case (∞), where all directions are characteristic; and we shall examine in detail case $(2_{10\rho})$, where interesting second-order resonance phenomena appear.

When $\Lambda = I$ the operator $L = L_{F_2,\Lambda}$ is given by

$$L(H) = \mathrm{Jac}(H)\cdot F_2 - \mathrm{Jac}(F_2)\cdot H\ .$$

• *Case* (∞).

In this case we have

$$L(u_{d,j}) = (d-2)u_{d+1,j+1} - v_{d+1,j} \qquad \text{and} \qquad L(v_{d,j}) = (d-1)v_{d+1,j+1}$$

for all $d \geq 2$ and $j = 0, \ldots, d$. Therefore

$$\mathrm{Im}\,L|_{\mathcal{H}^d} = \begin{cases} \mathrm{Span}\,\bigl(u_{d+1,2},\ldots,u_{d+1,d+1},(d-2)u_{d+1,1}-v_{d+1,0},v_{d+1,1},\ldots,v_{d+1,d+1}\bigr) & \text{for } d>2, \\ \mathrm{Span}\,(v_{3,0},\ldots,v_{3,3}) & \text{for } d=2. \end{cases}$$

Thus

$$(\mathrm{Im}\, L|_{\mathcal{H}^d})^\perp = \begin{cases} \mathrm{Span}\,(u_{d+1,0}, (d+1)u_{d+1,1} + (d-2)v_{d+1,0}) & \text{for } d>2, \\ \mathrm{Span}\,(u_{3,0},\ldots,u_{3,3}) & \text{for } d=2. \end{cases}$$

It then follows that every formal power series of the form

$$F(z,w) = (z + z^2 + O_3, w + zw + O_3)$$

is formally conjugated to a power series of the form

$$G(z,w) = \bigl(z + z^2 + a_0 z^3 + a_1 z^2 w + a_2 z w^2 + \varphi(w) + z\psi'(w), zw + w\psi'(w) - 3\psi(w)\bigr)$$

where $\varphi \in \mathbb{C}[\![\zeta]\!]$ is an arbitrary power series of order at least 3, $\psi \in \mathbb{C}[\![\zeta]\!]$ is an arbitrary power series of order at least 4 and a_0, a_1, $a_2 \in \mathbb{C}$.

• *Case* (1_{00}).
In this case we have

$$L(u_{d,j}) = (j-d)u_{d+1,j+2} + 2v_{d+1,j+1} \qquad \text{and} \qquad L(v_{d,j}) = (j-d)v_{d+1,j+2}$$

for all $d \geq 2$ and $j = 0,\ldots,d$. Therefore

$$\mathrm{Im}\, L|_{\mathcal{H}^d} = \mathrm{Span}\,(2v_{d+1,1} - du_{d+1,2}, u_{d+1,3},\ldots,u_{d+1,d+1}, v_{d+1,2},\ldots,v_{d+1,d+1})\ ,$$

and thus

$$(\mathrm{Im}\, L|_{\mathcal{H}^d})^\perp = \mathrm{Span}\,(u_{d+1,0}, u_{d+1,1}, v_{d+1,0}, u_{d+1,2} + v_{d+1,1})\ .$$

It then follows that every formal power series of the form

$$F(z,w) = (z + O_3, w - z^2 + O_3)$$

is formally conjugated to a power series of the form

$$G(z,w) = \bigl(z + w\varphi_1(w) + z\varphi_2(w) + z^2\psi(w), w - z^2 + w\varphi_3(w) + zw\psi(w)\bigr)\ ,$$

where φ_1, φ_2, $\varphi_3 \in \mathbb{C}[\![\zeta]\!]$ are arbitrary power series of order at least 2, and $\psi \in \mathbb{C}[\![\zeta]\!]$ is an arbitrary power series of order at least 1.

• *Case* (1_{10}).
In this case we have

$$L(u_{d,j}) = (2-d)u_{d+1,j+1} - (d-j)u_{d+1,j+2} + 2v_{d+1,j+1} + v_{d+1,j}$$

and

$$L(v_{d,j}) = (1-d)v_{d+1,j+1} - (d-j)v_{d+1,j+2}$$

for all $d \geq 2$ and $j = 0,\ldots,d$. Therefore

$$\mathrm{Im}\, L|_{\mathcal{H}^d} = \begin{cases} \mathrm{Span}\,((2-d)u_{d+1,1} + v_{d+1,0}, u_{d+1,2},\ldots,u_{d+1,d+1}, v_{d+1,1},\ldots,v_{d+1,d+1}) & \text{for } d>2, \\ \mathrm{Span}\,(v_{3,0} - 2u_{3,2}, u_{3,3}, v_{3,1}, v_{3,2}, v_{3,3}) & \text{for } d=2, \end{cases}$$

and thus

$$(\mathrm{Im}\, L|_{\mathcal{H}^d})^\perp = \begin{cases} \mathrm{Span}\,(u_{d+1,0}, (d+1)u_{d+1,1} + (d-2)v_{d+1,0}) & \text{for } d>2, \\ \mathrm{Span}\,(u_{3,0}, u_{3,1}, 3u_{3,2} + 2v_{3,0}) & \text{for } d=2. \end{cases}$$

It then follows that every formal power series of the form

$$F(z,w) = (z - z^2 + O_3, w - z^2 - zw + O_3)$$

is formally conjugated to a power series of the form

$$G(z,w) = \bigl(z - z^2 + \varphi(w) + a_1 z w^2 + 3a_2 z^2 w + z\psi'(w), w - z^2 - zw + 2a_2 w^3 + w\psi'(w) - 3\psi(w)\bigr)\ ,$$

where $\varphi \in \mathbb{C}[\![\zeta]\!]$ is an arbitrary power series of order at least 3, $\psi \in \mathbb{C}[\![\zeta]\!]$ is an arbitrary power series of order at least 4, and a_1, $a_2 \in \mathbb{C}$.

• *Case* (2_{001}).
In this case we have

$$L(u_{d,j})=(d-j)u_{d+1,j+1}-v_{d+1,j}\qquad\text{and}\qquad L(v_{d,j})=(d-j-1)v_{d+1,j+1}$$

for all $d\ge 2$ and $j=0,\dots,d$. It follows that

$$\operatorname{Im}L|_{\mathcal{H}^d}=\operatorname{Span}\left(du_{d+1,1}-v_{d+1,0},u_{d+1,2},\dots,u_{d+1,d},v_{d+1,1},\dots,v_{d+1,d+1}\right)$$

and hence

$$(\operatorname{Im}L|_{\mathcal{H}^d})^\perp=\operatorname{Span}\left(u_{d+1,0},u_{d+1,d+1},(d+1)u_{d+1,1}+dv_{d+1,0}\right)\ .$$

It then follows that every formal germ of the form

$$F(z,w)=(z+O_3,w+zw+O_3)$$

is formally conjugated to a germ of the form

$$G(z,w)=\big(z+\varphi_1(z)+\varphi_2(w)+z\psi'(w),zw+w\psi'(w)-\psi(w)\big)$$

where φ_1, φ_2, $\psi\in\mathbb{C}[\![\zeta]\!]$ are arbitrary power series of order at least 3.

• *Case* $(2_{10\rho})$.
In this case we have

$$L(u_{d,j})=(d-j-d\rho+2\rho)u_{d+1,j+1}+(\rho-1)v_{d+1,j}\quad\text{and}\quad L(v_{d,j})=(d-j-d\rho+\rho-1)v_{d+1,j+1}\tag{4.1}$$

for all $d\ge 2$ and $j=0,\dots,d$. Here we can shall see the resonance phenomena we mentioned at the beginning of this section: for some values of ρ the dimension of the kernel of $L|_{\mathcal{H}^d}$ can increase, and in some cases we shall end up with a normal form depending on power series evaluated in monomials of the form $z^{b-a}w^a$.

Let us put

$$E_d=\left\{\frac{d-j-1}{d-1}\;\middle|\;j=0,\dots,d\right\}\setminus\{0\}\quad\text{and}\quad F_d=\left\{\frac{d-j}{d-2}\;\middle|\;j=0,\dots,d-1\right\}$$

(we are excluding 0 because $\rho\neq 0$ by assumption), where E_d is defined for all $d\ge 2$ whereas F_d is defined for all $d\ge 3$, and set

$$\mathcal{E}=\bigcup_{d\ge 2}E_d=\big((0,1]\cap\mathbb{Q}\big)\cup\left\{-\frac{1}{n}\;\middle|\;n\in\mathbb{N}^*\right\}$$

and

$$\mathcal{F}=\bigcup_{d\ge 3}F_d=\big((0,1]\cap\mathbb{Q}\big)\cup\left\{1+\frac{1}{n},1+\frac{2}{n}\;\middle|\;n\in\mathbb{N}^*\right\}.$$

So $\mathcal{E}$ is the set of $\rho\in\mathbb{C}^*$ such that $L(v_{d,j})=0$ for some $d\ge 2$ and $0\le j\le d$, while $\mathcal{F}$ is the set of $\rho\in\mathbb{C}^*$ such that $L(u_{d,j})=(\rho-1)v_{d+1,j}$ for some $d\ge 3$ and $0\le j\le d-1$.

Let us first discuss the non-resonant case, when $\rho\notin\mathcal{E}\cup\mathcal{F}$. Then none of the coefficients in (4.1) vanishes, and thus

$$\operatorname{Im}L|_{\mathcal{H}^d}=\operatorname{Span}\left((d-d\rho+2\rho)u_{d+1,1}+(\rho-1)v_{d+1,0},u_{d+1,2},\dots,u_{d+1,d+1},v_{d+1,1},\dots,v_{d+1,d+1}\right)\ ,$$

and hence

$$(\operatorname{Im} L|_{\mathcal{H}^d})^\perp = \operatorname{Span}\left(u_{d+1,0}, (1-\rho)(d+1)u_{d+1,1} + (d(1-\rho)+2\rho)v_{d+1,0}\right) .$$

It then follows that every formal germ of the form

$$F(z,w) = (z - \rho z^2 + O_3, w + (1-\rho)zw + O_3)$$

with $\rho \notin \mathcal{E} \cup \mathcal{F}$ (and $\rho \neq 0$) is formally conjugated to a germ of the form

$$G(z,w) = \left(z - \rho z^2 + \varphi(w) + (1-\rho)z\psi'(w), w + (1-\rho)zw + (1-\rho)w\psi'(w) + (3\rho-1)\psi(z)\right) ,$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are arbitrary power series of order at least 3.

Assume now $\rho \in \mathcal{F} \setminus \mathcal{E}$. Then $L(v_{d,j}) \neq O$ always, and thus $v_{d+1,j} \in \operatorname{Im} L|_{\mathcal{H}^d}$ for all $d \geq 2$ and all $j = 1, \ldots, d+1$. Since $\rho > 1$, if $d > 2$ it also follows that $u_{d+1,j+1} \in \operatorname{Im} L|_{\mathcal{H}^d}$ for $j = 1, \ldots, d$.

Now, if $\rho = 1 + (1/n)$ then

$$\frac{d}{d-2} = \rho \quad \Longleftrightarrow \quad d = 2(n+1) ,$$

and

$$\frac{d-1}{d-2} = \rho \quad \Longleftrightarrow \quad d = n+2 .$$

Taking care of the case $d = 2$ separately, we then have

$$\operatorname{Im} L|_{\mathcal{H}^d} = \begin{cases} \operatorname{Span}\left((d - d\rho + 2\rho)u_{d+1,1} + (\rho-1)v_{d+1,0}, u_{d+1,2}, \ldots, u_{d+1,d+1}, v_{d+1,1}, \ldots, v_{d+1,d+1}\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } d \geq 3,\ d \neq n+2,\ 2(n+1), \\ \operatorname{Span}\left(u_{d+1,1} + (\rho-1)v_{d+1,0}, u_{d+1,3}, \ldots, u_{d+1,d+1}, v_{d+1,1}, \ldots, v_{d+1,d+1}\right) & \text{for } d = n+2, \\ \operatorname{Span}\left(u_{d+1,2}, \ldots, u_{d+1,d+1}, v_{d+1,0}, \ldots, v_{d+1,d+1}\right) & \text{for } d = 2(n+1), \\ \operatorname{Span}\left(2u_{3,1} + (\rho-1)v_{3,0}, u_{3,2}, v_{3,1}, v_{3,2}, v_{3,3}\right) & \text{for } d = 2, \end{cases}$$

and hence

$$(\operatorname{Im} L|_{\mathcal{H}^d})^\perp = \begin{cases} \operatorname{Span}\left(u_{d+1,0}, (1-\rho)(d+1)u_{d+1,1} + (d(1-\rho)+2\rho)v_{d+1,0}\right) & \text{for } d \geq 3,\ d \neq n+2,\ 2(n+1), \\ \operatorname{Span}\left(u_{d+1,0}, u_{d+1,2}, (1-\rho)(d+1)u_{d+1,1} + v_{d+1,0}\right) & \text{for } d = n+2, \\ \operatorname{Span}\left(u_{d+1,0}, u_{d+1,1}\right) & \text{for } d = 2(n+1), \\ \operatorname{Span}\left(u_{3,0}, u_{3,3}, 3(1-\rho)u_{3,1} + 2v_{3,0}\right) & \text{for } d = 2. \end{cases}$$

It then follows that every formal germ of the form

$$F(z,w) = \left(z - \left(1 + \frac{1}{n}\right)z^2 + O_3, w - \frac{1}{n}zw + O_3\right)$$

with $n \in \mathbb{N}^*$ is formally conjugated to a germ of the form

$$\begin{aligned} G(z,w) = \biggl(z - \left(1 + \frac{1}{n}\right)z^2 + \varphi(w) + (1-\rho)z\psi'(w) + a_0 z^3 + a_1 z^2 w^{n+1}, \\ w - \frac{1}{n}zw + (1-\rho)w\psi'(w) + (3\rho-1)\psi(w)\biggr) \ , \end{aligned}$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are arbitrary power series of order at least 3, and a_0, $a_1 \in \mathbb{C}$.

If instead $\rho = 1 + (2/m)$ with m odd (if m is even we are again in the previous case) then

$$\frac{d}{d-2} = \rho \quad \Longleftrightarrow \quad d = m + 2 ,$$

whereas $\frac{d-1}{d-2} \neq \rho$ always. Hence

$$\operatorname{Im} L|_{\mathcal{H}^d} = \begin{cases} \operatorname{Span}\left((d - d\rho + 2\rho)u_{d+1,1} + (\rho - 1)v_{d+1,0}, u_{d+1,2}, \ldots, u_{d+1,d+1}, v_{d+1,1}, \ldots, v_{d+1,d+1}\right) \\ \hfill \text{for } d \geq 3,\ d \neq m+2, \\ \operatorname{Span}\left(u_{d+1,2}, \ldots, u_{d+1,d+1}, v_{d+1,0}, \ldots, v_{d+1,d+1}\right) \hfill \text{for } d = m+2, \\ \operatorname{Span}\left(2u_{3,1} + (\rho - 1)v_{3,0}, u_{3,2}, v_{3,1}, v_{3,2}, v_{3,3}\right) \hfill \text{for } d = 2, \end{cases}$$

and thus

$$(\operatorname{Im} L|_{\mathcal{H}^d})^\perp = \begin{cases} \operatorname{Span}\left(u_{d+1,0}, (1-\rho)(d+1)u_{d+1,1} + (d - d\rho + 2\rho)v_{d+1,0}\right) & \text{for } d \geq 3,\ d \neq m+2, \\ \operatorname{Span}\left(u_{d+1,0}, u_{d+1,1}\right) & \text{for } d = m+2, \\ \operatorname{Span}\left(u_{3,0}, u_{3,3}, 3(1-\rho)u_{3,1} + 2v_{3,0}\right) & \text{for } d = 2. \end{cases}$$

It then follows that every formal germ of the form

$$F(z,w) = \left(z - \left(1 + \frac{2}{m}\right) z^2 + O_3, w - \frac{2}{m} zw + O_3\right)$$

with $m \in \mathbb{N}^*$ odd is formally conjugated to a germ of the form

$$\begin{aligned} G(z,w) = \biggl(z - \left(1 + \frac{2}{m}\right) z^2 + \varphi(w) + a_0 z^3 + (1-\rho) z(w\psi'(w) + \psi(w)), \\ w - \frac{2}{m} zw + (1-\rho) w^2 \psi'(w) + 2\rho w \psi(w)\biggr) \ , \end{aligned}$$

where $\varphi \in \mathbb{C}[\![\zeta]\!]$ is an arbitrary power series of order at least 3, $\psi \in \mathbb{C}[\![\zeta]\!]$ is an arbitrary power series of order at least 2, and a_0, $a_1 \in \mathbb{C}$.

Now let us consider the case $\rho = -1/n \in \mathcal{E} \setminus \mathcal{F}$. In this case the coefficients in the expression of $L(u_{d,j})$ are always different from zero (with the exception of $d = j = 2$), whereas

$$d - j - d\rho + \rho - 1 = 0 \quad \Longleftrightarrow \quad j = d = n + 1 .$$

It follows that

$$\operatorname{Im} L|_{\mathcal{H}^d} = \begin{cases} \operatorname{Span}\left((d - d\rho + 2\rho)u_{d+1,1} + (\rho - 1)v_{d+1,0}, u_{d+1,2}, \ldots, u_{d+1,d+1}, v_{d+1,1}, \ldots, v_{d+1,d+1}\right) \\ \hfill \text{for } d \geq 3,\ d \neq n+1, \\ \operatorname{Span}\left((d - d\rho + 2\rho)u_{d+1,1} + (\rho - 1)v_{d+1,0}, u_{d+1,2}, \ldots, u_{d+1,d+1}, v_{d+1,1}, \ldots, v_{d+1,d}\right) \\ \hfill \text{for } d = n+1, \\ \operatorname{Span}\left(2u_{3,1} + (\rho - 1)v_{3,0}, u_{3,2}, v_{3,1}, v_{3,2}, v_{3,3}\right) \hfill \text{for } d = 2, \end{cases}$$

and thus

$$(\operatorname{Im} L|_{\mathcal{H}^d})^\perp = \begin{cases} \operatorname{Span}\left(u_{d+1,0}, (1-\rho)(d+1)u_{d+1,1} + (d - d\rho + 2\rho)v_{d+1,0}\right) & \text{for } d \geq 3,\ d \neq n+1, \\ \operatorname{Span}\left(u_{d+1,0}, v_{d+1,d+1}, (1-\rho)(d+1)u_{d+1,1} + (d - d\rho + 2\rho)v_{d+1,0}\right) & \text{for } d = n+1, \\ \operatorname{Span}\left(u_{3,0}, u_{3,3}, 3(1-\rho)u_{3,1} + 2v_{3,0}\right) & \text{for } d = 2. \end{cases}$$

It then follows that every formal germ of the form

$$F(z,w)=\left(z+\frac{1}{n}z^2+O_3,w+\left(1+\frac{1}{n}\right)zw+O_3\right)$$

with $n\in\mathbb{N}^*$ is formally conjugated to a germ of the form

$$\begin{aligned}G(z,w)=\biggl(&z+\frac{1}{n}z^2+\varphi(w)+a_0z^3+(1-\rho)z(w\psi'(w)+\psi(w)),\\&w+\left(1+\frac{1}{n}\right)zw+\psi(z)+a_1z^{n+2}+(1-\rho)w^2\psi'(w)+2\rho w\psi(w)\biggr)\ ,\end{aligned}$$

where $\varphi\in\mathbb{C}[\![\zeta]\!]$ is an arbitrary power series of order at least 3, $\psi\in\mathbb{C}[\![\zeta]\!]$ is an arbitrary power series of order at least 2, and a_0, $a_1\in\mathbb{C}$.

Let us now discuss the extreme case $\rho=1$. It is clear that

$$\operatorname{Im}L|_{\mathcal{H}^d}=\operatorname{Span}\left(u_{d+1,1},u_{d+1,2},u_{d+1,4},\dots,u_{d+1,d+1},v_{d+1,2},\dots,v_{d+1,d+1}\right)\ ,$$

and hence

$$(\operatorname{Im}L|_{\mathcal{H}^d})^\perp=\operatorname{Span}\left(u_{d+1,0},u_{d+1,3},v_{d+1,0},v_{d+1,1}\right)\ ,$$

It then follows that every formal germ of the form

$$F(z,w)=\left(z-z^2+O_3,w+O_3\right)$$

is formally conjugated to a germ of the form

$$G(z,w)=\left(z-z^2+\varphi_1(w)+z^3\psi(w),w+\varphi_2(w)+z\varphi_3(w)\right)\ ,$$

where φ_1, $\varphi_2\in\mathbb{C}[\![\zeta]\!]$ are arbitrary power series of order at least 3, $\varphi_3\in\mathbb{C}[\![\zeta]\!]$ is an arbitrary power series of order at least 2, and $\psi\in\mathbb{C}[\![\zeta]\!]$ is an arbitrary power series.

We are left with the case $\rho\in(0,1)\cap\mathbb{Q}$. Write $\rho=a/b$ with a, $b\in\mathbb{N}$ coprime and $0<a<b$. Now

$$d-j-1-\frac{a}{b}(d-1)=0\quad\iff\quad j=\frac{(d-1)(b-a)}{b}\ ;$$

since a and b are coprime, this happens if and only if $d=b\ell+1$ and $j=(b-a)\ell$ for some $\ell\geq1$. Analogously,

$$d-j-\frac{a}{b}(d-2)=0\quad\iff\quad j=d-\frac{a(d-2)}{b}\ ;$$

again, being a and b coprime, this happens if and only if $d=b\ell+2$ and $j=(b-a)\ell+2$ for some $\ell\geq0$. It follows that

$$\operatorname{Im}L|_{\mathcal{H}^d}$$
$$=\begin{cases}\operatorname{Span}\left((d-d\rho+2\rho)u_{d+1,1}+(\rho-1)v_{d+1,0},u_{d+1,2},\dots,u_{d+1,d+1},v_{d+1,1},\dots,v_{d+1,d+1}\right)\\ \hfill\text{for } d\geq3,\ d\not\equiv1,\ 2 \pmod b\\ \operatorname{Span}\left((d-d\rho+2\rho)u_{d+1,1}+(\rho-1)v_{d+1,0},u_{d+1,2},\dots,\widehat{u_{d+1,(b-a)\ell+2}},\dots,u_{d+1,d+1},\right.\\ \qquad\left.v_{d+1,1},\dots,\widehat{v_{d+1,(b-a)\ell+1}},\dots,v_{d+1,d+1},\frac{a}{b}u_{d+1,(b-a)\ell+2}-\left(\frac{a}{b}-1\right)v_{d+1,(b-a)\ell+1}\right)\\ \hfill\text{for } d=b\ell+1,\\ \operatorname{Span}\left((d-d\rho+2\rho)u_{d+1,1}+(\rho-1)v_{d+1,0},u_{d+1,2},\dots,\widehat{u_{d+1,(b-a)\ell+3}},\dots,u_{d+1,d+1},\right.\\ \qquad\left.v_{d+1,1},\dots,v_{d+1,d+1}\right)\hfill\text{for } d=b\ell+2,\\ \operatorname{Span}\left(2u_{3,1}+(\rho-1)v_{3,0},u_{3,2},v_{3,1},v_{3,2},v_{3,3}\right)\hfill\text{for } d=2,\end{cases}$$

(where the hat indicates that that term is missing from the list), and thus

$$
\begin{aligned}
&(\operatorname{Im} L|_{\mathcal{H}^d})^\perp \\
&= \begin{cases}
\operatorname{Span}(u_{d+1,0}, (1-\rho)u_{d+1,1} + (d - d\rho + 2\rho)v_{d+1,0}) & \text{for } d \geq 3,\ d \not\equiv 1,\ 2 \pmod b, \\
\operatorname{Span}(u_{d+1,0}, (1-\rho)u_{d+1,1} + (d - d\rho + 2\rho)v_{d+1,0}, & \\
\qquad (b-a)(a\ell+1)u_{d+1,(b-a)\ell+2} + a\big((b-a)\ell+2\big)v_{d+1,(b-a)\ell+1}) & \text{for } d = b\ell+1, \\
\operatorname{Span}\big(u_{d+1,0}, u_{d+1,(b-a)\ell+3}, (1-\rho)u_{d+1,1} + (d - d\rho + 2\rho)v_{d+1,0}\big) & \text{for } d = b\ell+2, \\
\operatorname{Span}(u_{3,0}, u_{3,3}, 3(1-\rho)u_{3,1} + 2v_{3,0}) & \text{for } d = 2.
\end{cases}
\end{aligned}
$$

It then follows that every formal germ of the form

$$
F(z,w) = \left(z - \frac{a}{b}z^2 + O_3, w + \left(1 - \frac{a}{b}\right) zw + O_3\right)
$$

with $a/b \in (0,1) \cap \mathbb{Q}$ and a, b coprime, is formally conjugated to a germ of the form

$$
\begin{aligned}
&G(z,w) \\
&= \Bigg(z - \frac{a}{b}z^2 + \varphi(w) + z^3\varphi_0(z^{b-a}w^a) + (b-a)\frac{\partial}{\partial w}\big(z^2 w\chi(z^{b-a}w^a)\big) + \left(1 - \frac{a}{b}\right) z(w\psi'(w) + \psi(w)), \\
&\qquad w + \left(1 - \frac{a}{b}\right) zw + a\frac{\partial}{\partial z}\big(z^2 w\chi(z^{b-a}w^a)\big) + \left(1 - \frac{a}{b}\right) w^2\psi'(w) + 2\frac{a}{b} w\psi(w)\Bigg)\ ,
\end{aligned}
$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are arbitrary power series of order at least 3, and φ_0, $\chi \in \mathbb{C}[\![\zeta]\!]$ are arbitrary power series of order at least 1.

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Formal Poincaré-Dulac renormalization for holomorphic germs

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Abstract. Applying a general renormalization procedure for formal self-maps, producing a formal normal form simpler than the classical Poincaré-Dulac normal form, we shall give a complete list of normal forms for bi-dimensional superattracting germs with non-vanishing quadratic term; in most cases, our normal forms will be the simplest possible ones (in the sense of Wang, Zheng and Peng). We shall also discuss a few examples of renormalization of germs tangent to the identity, revealing interesting second-order resonance phenomena.

0. Introduction

In the study of a class of holomorphic dynamical systems, an important goal often is the classification under topological, holomorphic or formal conjugation. In particular, for each dynamical system in the class one would like to have a definite way of choosing a (hopefully simpler, possibly unique) representative in the same conjugacy class; a *normal form* of the original dynamical system.

The formal classification of one-dimensional germs is well-known (see, e.g., [A2]): if

$$f(z) = \lambda z + a_\mu z^\mu + O_{\mu+1} \in \mathbb{C}[\![z]\!]$$

is a one-dimensional formal power series with complex cofficients and vanishing constant term, where $a_\mu \neq 0$ and $O_{\mu+1}$ is a remainder term of order at least $\mu + 1$, then f is formally conjugated to:

- $g(z) = \lambda z$ if $\lambda \neq 0$ and λ is not a root of unity;
- $g(z) = z^\mu$ if $\lambda = 0$; and to
- $g(z) = \lambda z - z^{nq+1} + \alpha z^{2nq+1}$ if λ is a primitive q-th root of unity, for suitable $n \geq 1$ and $\alpha \in \mathbb{C}$ that are formal invariant (and $q = 1$ and $n = \mu$ when $\lambda = 1$).

In several variables, the most famous kind of normal form for local holomorphic dynamical systems (i.e., germs of holomorphic vector fields at a singular point, or germs of holomorphic self-maps with a fixed point) is the *Poincaré-Dulac normal form* with respect to formal conjugation; let us recall very quickly its definition, at least in the setting we are interested here, that is of formal self-maps with a fixed point, that we can assume to be the origin in $\mathbb{C}^n$, without discussing here convergence issues.

So let $F \in \widehat{\mathcal{O}}^n$ be a formal transformation in n complex variables, where $\widehat{\mathcal{O}}^n$ denotes the space of n-tuples of power series in n variables with vanishing constant term, and let Λ denote the (not necessarily invertible) linear term of F; up to a linear change of variables, we can assume that Λ is in Jordan normal form. For simplicity, given a linear map $\Lambda \in M_{n,n}(\mathbb{C})$ we shall denote by $\widehat{\mathcal{O}}^n_\Lambda$ the set of formal transformations in $\widehat{\mathcal{O}}^n$ with Λ as linear part. If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues

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of Λ, we shall say that a multi-index $Q=(q_1,\ldots,q_n)\in\mathbb{N}^n$ with $q_1+\cdots+q_n\ge 2$ is *Λ-resonant* if there is $j\in\{1,\ldots,n\}$ such that $\lambda_1^{q_1}\cdots\lambda_n^{q_n}=\lambda_j$. If this happens, we shall say that the monomial $z_1^{q_1}\cdots z_n^{q_n}e_j$ is *Λ-resonant,* where $\{e_1,\ldots,e_n\}$ is the canonical basis of $\mathbb{C}^n$. Then (see, e.g., [Ar], [R1, 2], [Rü]) given $F\in\widehat{\mathcal{O}}^n_\Lambda$ it is possible to find a (not unique, in general) invertible formal transformation $\Phi\in\widehat{\mathcal{O}}^n_I$ with identity linear part such that $G=\Phi^{-1}\circ F\circ\Phi$ contains only Λ-resonant monomials.

The formal transformation G is a *Poincaré-Dulac normal form* of F; notice that, since $\Phi\in\widehat{\mathcal{O}}^n_I$, the linear part of G is still Λ. More generally, we shall say that a $G\in\widehat{\mathcal{O}}^n_\Lambda$ is *in Poincaré-Dulac normal form* if G contains only Λ-resonant monomials.

The importance of this result cannot be underestimated, and it has been applied uncountably many times; however it has some limitations. For instance, if $\Lambda=O$ or $\Lambda=I$ then *all* monomials are resonant; and thus in these cases any $F\in\widehat{\mathcal{O}}^n_\Lambda$ is in Poincaré-Dulac normal form, and a further simplification (a *renormalization*) is necessary. Actually, even when a Poincaré-Dulac normal form is different from the original germ, it is often possible to further simplify the germ by applying invertible transformations preserving the property of being in Poincaré-Dulac normal form.

This idea of renormalizing Poincaré-Dulac normal forms is not new in the context of vector fields; see, e.g., [AFGG, B1, BS, G, KOW, LS, Mu1, Mu2] and references therein. On the other hand, with a few exceptions (see, for instance, [B2, CD]) this idea has been exploited in the context of self-maps only recently. One example is [AT1], where it is applied to a particular class of self-maps with identity linear part. More important for our aims are [WZP1, 2], where the authors, following [KOW], construct an a priori infinite sequence of renormalizations giving simpler and simpler normal forms.

Let us roughly describe the main ideas. For each $\nu\ge 2$ let $\mathcal{H}^\nu$ denote the space of n-tuples of homogeneous polynomials in n variables of degree ν. Then every $F\in\widehat{\mathcal{O}}^n_\Lambda$ admits a *homogeneous expansion*

$$F=\Lambda+\sum_{\nu\ge 2}F_\nu\;,$$

where $F_\nu\in\mathcal{H}^\nu$ is the *ν-homogeneous term* of F. We shall also use the notation $\{G\}_\nu$ to denote the ν-homogeneous term of a formal transformation G.

If $\Phi=I+\sum_{\nu\ge 2}H_\nu\in\widehat{\mathcal{O}}^n_I$ is the homogeneous expansion of an invertible formal transformation, then it turns out that, if $L_\Lambda\colon\widehat{\mathcal{O}}^n\to\widehat{\mathcal{O}}^n$ is defined by setting $L_\Lambda(H)=H\circ\Lambda-\Lambda H$, then $L_\Lambda(\mathcal{H}^\nu)\subseteq\mathcal{H}^\nu$ and

$$\{\Phi^{-1}\circ F\circ\Phi\}_\nu=F_\nu-L_\Lambda(H_\nu)+R_\nu \tag{0.1}$$

for all $\nu\ge 2$, where R_ν is a remainder term depending only on F_ρ and H_σ with ρ, $\sigma<\nu$. This suggests to consider for each $\nu\ge 2$ splittings of the form

$$\mathcal{H}^\nu=\operatorname{Im}L^\nu_\Lambda\oplus\mathcal{N}^\nu\quad\text{and}\quad\mathcal{H}^\nu=\operatorname{Ker}L^\nu_\Lambda\oplus\mathcal{M}^\nu$$

where $L^\nu_\Lambda=L_\Lambda|_{\mathcal{H}^\nu}$, and $\mathcal{N}^\nu$ and $\mathcal{M}^\nu$ are suitable complementary subspaces. Then (0.1) implies that we can inductively choose $H_\nu\in\mathcal{M}^\nu$ so that $\{\Phi^{-1}\circ F\circ\Phi\}_\nu\in\mathcal{N}^\nu$ for all $\nu\ge 2$; we shall say that $G=\Phi^{-1}\circ F\circ\Phi$ is a *first order normal form* of F (with respect to the chosen complementary subspaces). Furthermore, it is not difficult to see that the quadratic (actually, the first non-linear non-vanishing) homogeneous term of G is a formal invariant, that is it is the same for all first order normal forms of F. Notice that when $\Lambda=O$ or $\Lambda=I$ we have $L_\Lambda\equiv O$, and thus in these cases every $F\in\widehat{\mathcal{O}}^n_\Lambda$ is a first order normal form.

When Λ is diagonal, $\operatorname{Ker}L_\Lambda$ is generated by the resonant monomials, and $\operatorname{Im}L_\Lambda$ is generated by the non-resonant monomials. Furthermore, for each $\nu\ge 2$ we have the splitting

$\mathcal{H}^\nu = \operatorname{Im} L^\nu_\Lambda \oplus \operatorname{Ker} L^\nu_\Lambda$, and thus taking $\mathcal{N}^\mu = \operatorname{Ker} L^\nu_\Lambda$ and $\mathcal{M}^\mu = \operatorname{Im} L^\nu_\Lambda$ we have recovered the classical Poincaré-Dulac normal form (when Λ has a nilpotent part the situation is only slightly more complicated; see [Mu1, Section 4.5] for details).

Summing up, a Poincaré-Dulac formal normal form is composed by homogeneous terms contained in a complementary space of the image of the operator L_Λ. Furthermore, the quadratic homogeneous term is uniquely determined, and we can still act on the normal form by transformations having all homogeneous terms in the kernel of L_Λ.

The k-th renormalization follows the same pattern. Assume that F is in $(k-1)$-th normal form. Then there is a suitable (not necessarily linear if $k \geq 3$; see [WZP2] for details) operator $\mathcal{L}^k$, depending on the first k homogeneous terms of F, so that we can bring F in a normal form G whose all homogeneous terms belong to a chosen complementary subspace* of the image of $\mathcal{L}^k$, and the first $k+1$ homogeneous terms of G are uniquely determined; we shall say that G is in *k-th order normal form* (with respect to the chosen subspaces).

A formal transformation G is in *infinite order normal form* if it is in k-th normal form for all k, with respect to some choice of complementary subspaces and using the operators $\mathcal{L}^k$ defined using the first k homogeneous terms of G. The main result of [WZP2] then states that every element of $\widehat{\mathcal{O}}^n_\Lambda$ can be brought to a (possibly not unique) infinite order normal form by a sequence of formal conjugations tangent to the identity.

To apply these results, we need a rule for choosing complementary subspaces. It turns out that an efficient way of doing this is by taking orthogonal complements with respect to the Fischer Hermitian product, defined by (see [F])

$$\langle z_1^{p_1}\cdots z_n^{p_n} e_h, z_1^{q_1}\cdots z_n^{q_n} e_k\rangle = \begin{cases} 0 & \text{if } h \neq k \text{ or } p_j \neq q_j \text{ for some } j;\\ \dfrac{p_1!\cdots p_n!}{(p_1+\cdots+p_n)!} & \text{if } h = k \text{ and } p_j = q_j \text{ for all } j.\end{cases} \tag{0.2}$$

With this choice, as we shall see in Sections 2 and 3, the expression of the second order (and often infinite order) normal forms can be quite simple. For instance, in Section 2 we shall apply this procedure to the case of superattracting (i.e., with $\Lambda = O$) 2-dimensional formal transformations, case that has no analogue in the vector field setting, proving the following

Theorem 0.1: *Let $F \in \widehat{\mathcal{O}}^2_O$ be of the form $F(z,w) = F_2(z,w) + O_3$. Then:*

(i) *if $F_2(z,w) = (z^2, zw)$ or $F_2(z,w) = (-z^2, -z^2 - zw)$ then F is formally conjugated to an unique infinite order normal form*

$$G(z,w) = F_2(z,w) + \big(\varphi(w) - z\psi'(w), 2\psi(w)\big)\;,$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3;

(ii) *if $F_2(z,w) = (-zw, -z^2 - w^2)$ then F is formally conjugated to an unique infinite order normal form*

$$G(z,w) = F_2(z,w) + \big(-2\varphi(z+w) + 2\psi(w-z), \varphi(z+w) + \psi(w-z)\big)\;,$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3;

(iii) *if $F_2(z,w) = (zw, zw + w^2)$ then F is formally conjugated to an unique infinite order normal form*

$$G(z,w) = F_2(z,w) + \big(w\varphi'(z) + \psi(z), 2\varphi(z) - w\varphi'(z) - \psi(z)\big)\;,$$

* When $k \geq 3$ one has to choose a complementary subspace to a vector space of maximal dimension contained in the image of $\mathcal{L}^k_\nu$. Actually, [WZP2] talks of "the" subspace of maximal dimension contained in $\mathcal{L}^k_\nu$, but a priori it might not be unique.

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3;

(iv) *if $F_2(z,w) = \bigl(-\rho z^2, (1-\rho)zw\bigr)$ with $\rho \neq 0, 1$ then F is formally conjugated to an unique infinite order normal form*

$$G(z,w) = F_2(z,w) + \bigl((\rho-1)z\varphi'(w) + \psi(w), -2\rho\varphi(z)\bigr) ,$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3;

(v) *if $F_2(z,w) = (-z^2 + zw, w^2\bigr)$ then F is formally conjugated to an unique infinite order normal form*

$$G(z,w) = F_2(z,w) + \bigl(\varphi(\tfrac{z}{2} + w), -\tfrac{1}{4}\varphi(\tfrac{z}{2} + w) + \psi(z)\bigr) ,$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3;

(vi) *if $F_2(z,w) = \bigl(\rho z^2 + zw, (1+\rho)zw + w^2\bigr)$ with $\rho \neq 0, -1$ then F is formally conjugated to an unique infinite order normal form*

$$\begin{aligned} G(z,w) = F_2(z,w) + \Biggl(&\frac{1}{\rho}\left[\frac{1-\sqrt{-\rho}}{2m_\rho^2}\varphi(m_\rho z + w) + \frac{1+\sqrt{-\rho}}{2n_\rho^2}\varphi(n_\rho z + w)\right] \\ &+ \frac{1+\rho}{2\sqrt{-\rho}}\left(\frac{1}{m_\rho^2}\psi(m_\rho z + w) - \frac{1}{n_\rho^2}\psi(n_\rho z + w)\right), \\ &\frac{1-\sqrt{-\rho}}{2}\varphi(m_\rho z + w) + \frac{1+\sqrt{-\rho}}{2}\varphi(n_\rho z + w) \\ &+ \frac{\rho(1+\rho)}{2\sqrt{-\rho}}\bigl(\psi(m_\rho z + w) - \psi(n_\rho z + w)\bigr)\Biggr) \end{aligned}$$

where $\sqrt{-\rho}$ is any square root of $-\rho$,

$$m_\rho = \frac{\sqrt{-\rho} - \rho}{\rho(1+\rho)} , \quad n_\rho = -\frac{\sqrt{-\rho} + \rho}{\rho(1+\rho)} ,$$

and φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3;

(vii) *if $F_2(z,w) = \bigl(\rho(-z^2 + zw), (1-\rho)(zw - w^2)\bigr)$ with $\rho \neq 0, 1$ then F is formally conjugated to an unique infinite order normal form*

$$\begin{aligned} G(z,w) = F_2(z,w) + \Biggl(&z\frac{\partial}{\partial z}\bigl[\varphi(z+w) + \psi(z+w)\bigr] - \varphi(z+w), \\ &\frac{\rho-1}{\rho}\left(z\frac{\partial}{\partial z}\bigl[\varphi(z+w) - \psi(z+w)\bigr] - 3\varphi(z+w) + 2\psi(z+w)\right)\Biggr) \end{aligned}$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3;

(ix) *if $F_2(z,w) = (-z^2, -w^2)$ then F is formally conjugated to an unique infinite order normal form*

$$G(z,w) = F_2(z,w) + \bigl(\varphi(w), \psi(z)\bigr)$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3;

(x) *if $F_2(z,w) = (-\rho z^2, (1-\rho)zw - w^2)$ with $\rho \neq 0, 1$ then F is formally conjugated to an unique infinite order normal form*

$$G(z,w) = F_2(z,w) + \left(\varphi(w) + \frac{(1-\rho)^2}{4\rho}\psi\left(\frac{2}{1-\rho}z + w\right), \psi\left(\frac{2}{1-\rho}z + w\right)\right)$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3;

(xi) *if $F_2(z,w) = (-\rho z^2 + (1-\tau)zw, (1-\rho)zw - \tau w^2)$ with ρ, $\tau \neq 0$, 1 and $\rho + \tau \neq 1$ then F is formally conjugated to an unique infinite order normal form*

$$\begin{aligned} G(z,w) = F_2(z,w) + \Bigg(\frac{\tau}{\rho} \Bigg[& \frac{\sqrt{\rho+\tau-1}+\sqrt{\rho\tau}}{2m_{\rho,\tau}^2}\varphi(m_{\rho,\tau}z+w) \\ & + \frac{\sqrt{\rho+\tau-1}-\sqrt{\rho\tau}}{2n_{\rho,\tau}^2}\varphi(n_{\rho,\tau}z+w) \\ & + \frac{1}{m_{\rho,\tau}^2}\psi(m_{\rho,\tau}z+w) - \frac{1}{n_{\rho,\tau}^2}\psi(n_{\rho,\tau}z+w) \Bigg], \\ & \frac{\sqrt{\rho+\tau-1}+\sqrt{\rho\tau}}{2}\varphi(m_{\rho,\tau}z+w) \\ & + \frac{\sqrt{\rho+\tau-1}-\sqrt{\rho\tau}}{2}\varphi(n_{\rho,\tau}z+w) \\ & + \psi(m_{\rho,\tau}z+w) - \psi(n_{\rho,\tau}z+w) \Bigg), \end{aligned}$$

where

$$m_{\rho,\tau} = \frac{\sqrt{\rho\tau}\sqrt{\rho+\tau-1}-\rho\tau}{\rho(\rho-1)}, \qquad n_{\rho,\tau} = -\frac{\sqrt{\rho\tau}\sqrt{\rho+\tau-1}+\rho\tau}{\rho(\rho-1)}.$$

and φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3.

In [A1] we showed that the list of quadratic terms in this theorem gives a complete list of all possible quadratic terms up to linear change of coordinates, with the exception of four degenerate cases where one of the coordinates is identically zero. In these cases missing we shall anyway be able to give a second order normal form:

Proposition 0.2: *Let $F \in \widehat{\mathcal{O}}_O^2$ be of the form $F(z,w) = F_2(z,w) + O_3$. Then:*

(i) *if $F_2(z,w) = (0, -z^2)$ then F is formally conjugated to a unique second order normal form*

$$G(z,w) = F_2(z,w) + \big(\Phi(z,w), \psi(w)\big),$$

where $\psi \in \mathbb{C}[\![\zeta]\!]$ and $\Phi \in \mathbb{C}[\![z,w]\!]$ are power series of order at least 3;

(ii) *if $F_2(z,w) = (0, zw)$ then F is formally conjugated to a unique second order normal form*

$$G(z,w) = F_2(z,w) + \big(\Phi(z,w), 0\big),$$

where $\Phi \in \mathbb{C}[\![z,w]\!]$ is a power series of order at least 3;

(iii) *if $F_2(z,w) = (-z^2, 0)$ then F is formally conjugated to a unique second order normal form*

$$G(z,w) = F_2(z,w) + \big(\psi(w), \Phi(z,w)\big),$$

where $\psi \in \mathbb{C}[\![\zeta]\!]$ and $\Phi \in \mathbb{C}[\![z,w]\!]$ are power series of order at least 3;

(iv) *if $F_2(z,w) = (z^2 - zw, 0)$ then F is formally conjugated to a unique second order normal form*

$$G(z,w) = F_2(z,w) + \big(0, \Phi(z,w)\big),$$

where $\Phi \in \mathbb{C}[\![z,w]\!]$ is a power series of order at least 3.

Finally, in Section 3 we shall also discuss a few interesting examples with $\Lambda = I$, showing in particular the appearance of non-trivial second-order resonance phenomena. For instance, we shall prove the following

Proposition 0.3: *Let $F \in \widehat{\mathcal{O}}_I^2$ be of the form $F(z,w) = (z,w) + F_2(z,w) + O_3$, with*
$$F_2(z,w) = \bigl(-\rho z^2, (1-\rho)zw\bigr)$$
and $\rho \neq 0$. Put
$$\mathcal{E} = \bigl([0,1]\cap\mathbb{Q}\bigr) \cup \left\{-\frac{1}{n} \;\middle|\; n \in \mathbb{N}^*\right\} \quad \text{and} \quad \mathcal{F} = \bigl([0,1]\cap\mathbb{Q}\bigr) \cup \left\{1+\frac{1}{n}, 1+\frac{2}{n} \;\middle|\; n \in \mathbb{N}^*\right\} .$$
Then:

(i) *if $\rho \notin \mathcal{E}\cup\mathcal{F}$ then F is formally conjugated to a unique second order normal form*
$$G(z,w) = (z,w) + F_2(z,w) + \bigl(az^3 + \varphi(w) + (1-\rho)z\psi'(w), (1-\rho)w\psi'(w) + (3\rho-1)\psi(z)\bigr) ,$$
where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3, and $a \in \mathbb{C}$;

(ii) *if $\rho = 1+\frac{1}{n} \in \mathcal{F}\setminus\mathcal{E}$ then F is formally conjugated to a unique second order normal form*
$$\begin{aligned} G(z,w) = {}&(z,w) + F_2(z,w) \\ &+ \left(a_0z^3 + a_1z^2w^{n+1} + \varphi(w) - \frac{1}{n}z\psi'(w), -\frac{1}{n}w\psi'(w) + \left(2+\frac{3}{n}\right)\psi(w)\right) ,\end{aligned}$$
where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3, and a_0, $a_1 \in \mathbb{C}$;

(iii) *if $\rho = 1+\frac{2}{m} \in \mathcal{F}\setminus\mathcal{E}$ with m odd then F is formally conjugated to a unique second order normal form*
$$\begin{aligned} G(z,w) = {}&(z,w) + F_2(z,w) \\ &+ \left(a_0z^3 + \varphi(w) - \frac{2}{m}z\bigl(w\psi'(w) + \psi(w)\bigr), \right. \\ &\left. -\frac{2}{m}w^2\psi'(w) + \left(2+\frac{4}{m}\right)w\psi(w)\right) ,\end{aligned}$$
where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least respectively 3 and 2, and $a_0 \in \mathbb{C}$;

(iv) *if $\rho = -\frac{1}{n} \in \mathcal{E}\setminus\mathcal{F}$ then F is formally conjugated to a unique second order normal form*
$$\begin{aligned} G(z,w) = {}&(z,w) + F_2(z,w) \\ &+ \left(a_0z^3 + \varphi(w) + \left(1+\frac{1}{n}\right)z\bigl(w\psi'(w) + \psi(w)\bigr), \right. \\ &\left. a_1z^{n+2} + \psi(z) + \left(1+\frac{1}{n}\right)w^2\psi'(w) - \frac{2}{n}w\psi(w)\right) ,\end{aligned}$$
where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least respectively 3 and 2, and a_0 $a_1 \in \mathbb{C}$;

(v) *if $\rho = 1 \in \mathcal{E}\cap\mathcal{F}$ then F is formally conjugated to a unique second order normal form*
$$G(z,w) = (z,w) + F_2(z,w) + \bigl(\varphi_1(w) + z^3\psi(w), \varphi_2(w) + z\varphi_3(w)\bigr) ,$$
where φ_1, $\varphi_2 \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3, $\varphi_2 \in \mathbb{C}[\![\zeta]\!]$ is a power series of order at least 2, and $\varphi_3 \in \mathbb{C}[\![\zeta]\!]$ is a power series;

(vi) *if $\rho = a/b \in (0,1)\cap\mathbb{Q} \subset \mathcal{E}\setminus\mathcal{F}$ then F is formally conjugated to a unique second order normal form*
$$\begin{aligned} G(z,w) = {}&(z,w) + F_2(z,w) \\ &+\left(\varphi(w) + z^3\varphi_0(z^{b-a}w^a) + (b-a)\frac{\partial}{\partial w}\bigl(z^2w\chi(z^{b-a}w^a)\bigr) + \left(1-\frac{a}{b}\right)z\bigl(w\psi'(w) + \psi(w)\bigr), \right. \\ &\left. a\frac{\partial}{\partial z}\bigl(z^2w\chi(z^{b-a}w^a)\bigr) + \left(1-\frac{a}{b}\right)w^2\psi'(w) + 2\frac{a}{b}w\psi(w)\right) ,\end{aligned}$$
where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3, and φ_0, $\chi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 1.

1. Renormalization

In this section we shall recover, with a different proof, the part of the renormalization procedure of [WZP2] useful for our aims. One difference between our approach and theirs is that we shall systematically use the relations between homogeneous polynomials and symmetric multilinear maps instead of relying on higher order derivatives as in [WZP2].

Let us start collecting a few results on homogeneous polynomials and maps we shall need later.

Definition 1.1: We shall denote by $\mathcal{H}^d$ the space of *homogenous maps of degree* d, i.e., of n-tuples of homogeneous polynomials of degree $d \geq 1$ in the variables $(z_1, \ldots, z_n)$. It is well known (see, e.g., [C, pp. 79–88]) that to each $P \in \mathcal{H}^d$ is associated a *unique* symmetric multilinear map $\tilde{P}\colon (\mathbb{C}^n)^d \to \mathbb{C}^n$ such that

$$P(z) = \tilde{P}(z, \ldots, z)$$

for all $z \in \mathbb{C}^n$. We also set $\mathcal{H} = \prod_{d \geq 2} \mathcal{H}^d$.

Roughly speaking, the symmetric multilinear map associated to a homogeneous map H encodes the derivatives of H. For instance, it is easy to check that for each $H \in \mathcal{H}^d$ we have

$$(\operatorname{Jac} H)(z) \cdot v = d\, \tilde{H}(v, z, \ldots, z) \tag{1.1}$$

for all z, $v \in \mathbb{C}^n$.

Later on we shall need to compute the multilinear map associated to a homogeneous map obtained as a composition. The formula we are interested in is contained in the next lemma.

Lemma 1.1: *Assume that $P \in \mathcal{H}^d$ is of the form*

$$P(z) = \tilde{K}\big(H_{d_1}(z), \ldots, H_{d_r}(z)\big) \ ,$$

where $\tilde{K}$ is r-multilinear, $d_1 + \cdots + d_r = d$, and $H_{d_j} \in \mathcal{H}^{d_j}$ for $j = 1, \ldots, r$. Then

$$\tilde{P}(v, w, \ldots, w) = \frac{1}{d} \sum_{j=1}^{r} d_j \tilde{K}\big(H_{d_1}(w), \ldots, \tilde{H}_{d_j}(v, w, \ldots, w), \ldots, H_{d_r}(w)\big)$$

for all v, $w \in \mathbb{C}^n$.

Proof: Write $z = w + \varepsilon v$. Then

$$\begin{aligned} P(w) &+ d\varepsilon \tilde{P}(v, w, \ldots, w) + O(\varepsilon^2) \\ &= P(w + \varepsilon v) = \tilde{K}\big(\tilde{H}_{d_1}(w + \varepsilon v, \ldots, w + \varepsilon v), \ldots, \tilde{H}_{d_r}(w + \varepsilon v, \ldots, w + \varepsilon v)\big) \\ &= \tilde{K}\big(H_{d_1}(w), \ldots, H_{d_r}(w)\big) + \varepsilon \sum_{j=1}^{r} d_j \tilde{K}\big(H_{d_1}(w), \ldots, \tilde{H}_{d_j}(v, w, \ldots, w), \ldots, H_{d_r}(w)\big) + O(\varepsilon^2) \ , \end{aligned}$$

and we are done. □

Definition 1.2: Given a linear map $\Lambda \in M_{n,n}(\mathbb{C})$, we define a linear operator $L_\Lambda\colon \mathcal{H} \to \mathcal{H}$ by setting

$$L_\Lambda(H) = H \circ \Lambda - \Lambda H \ .$$

We shall say that a homogeneous map $H \in \mathcal{H}^d$ is Λ-*resonant* if $L_\Lambda(H) = O$, and we shall denote by $\mathcal{H}^d_\Lambda = \operatorname{Ker} L_\Lambda \cap \mathcal{H}^d$ the subspace of Λ-resonant homogeneous maps of degree d. Finally, we set $\mathcal{H}_\Lambda = \prod_{d \geq 2} \mathcal{H}^d_\Lambda$.

When Λ is diagonal, then the Λ-resonant monomials are exactly the resonant monomials appearing in the classical Poincaré-Dulac theory.

Definition 1.3: If $Q = (q_1, \ldots, q_n) \in \mathbb{N}^n$ is a multi-index and $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, we shall put $z^Q = z_1^{q_1} \cdots z_n^{q_n}$. Given $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \in M_{n,n}(\mathbb{C})$, we shall say that $Q \in \mathbb{N}^n$ with $q_1 + \cdots + q_n \geq 2$ is *Λ-resonant on the j-th coordinate* if $\lambda_1^{q_1} \cdots \lambda_n^{q_n} = \lambda_j$. If Q is Λ-resonant on the j-th coordinate, we shall also say that the monomial $z^Q e_j$ is *Λ-resonant*, where $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{C}^n$.

Remark 1.1: If $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \in M_{n,n}(\mathbb{C})$ is diagonal, and $z^Q e_j \in \mathcal{H}^d$ is a homogeneous monomial (with $q_1 + \cdots + q_n = d$), then (identifying the matrix Λ with the vector, still denoted by Λ, of its diagonal entries) we have

$$L_\Lambda(z^Q e_j) = (\Lambda^Q - \lambda_j) z^Q e_j \ .$$

Therefore $z^Q e_j$ is Λ-resonant if and only if Q is Λ-resonant in the j-th coordinate. In particular, a basis of $\mathcal{H}^d_\Lambda$ is given by the Λ-resonant monomials, and we have

$$\mathcal{H}^d = \mathcal{H}^d_\Lambda \oplus \mathrm{Im}\, L_\Lambda|_{\mathcal{H}^d}$$

for all $d \geq 2$.

It is possible to detect the Λ-resonance by using the associated multilinear map:

Lemma 1.2: *If $\Lambda \in M_{n,n}(\mathbb{C})$ and $H \in \mathcal{H}^d$ then H is Λ-resonant if and only if*

$$\tilde{H}(\Lambda v_1, \ldots, \Lambda v_d) = \Lambda \tilde{H}(v_1, \ldots, v_d) \tag{1.2}$$

for all $v_1, \ldots, v_d \in \mathbb{C}^n$. In particular, if $H \in \mathcal{H}^d_\Lambda$ then

$$\big((\mathrm{Jac}\, H) \circ \Lambda\big) \cdot \Lambda = \Lambda \cdot (\mathrm{Jac}\, H) \ . \tag{1.3}$$

Proof: One direction is trivial. Conversely, assume $H \in \mathcal{H}^d_\Lambda$. By definition, H is Λ-resonant if and only if $\tilde{H}(\Lambda w, \ldots, \Lambda w) = \Lambda \tilde{H}(w, \ldots, w)$ for all $w \in \mathbb{C}^n$. Put $w = z + \varepsilon v_1$; then

$$\begin{aligned} \tilde{H}(\Lambda z, \ldots, \Lambda z) + \varepsilon d\, \tilde{H}(\Lambda v_1, \Lambda z, \ldots, \Lambda z) + O(\varepsilon^2) &= \tilde{H}\big(\Lambda(z + \varepsilon v_1), \ldots, \Lambda(z + \varepsilon v_1)\big) \\ &= \Lambda \tilde{H}(z + \varepsilon v_1, \ldots, z + \varepsilon v_1) \\ &= \Lambda \tilde{H}(z, \ldots, z) + \varepsilon d\, \Lambda \tilde{H}(v_1, z, \ldots, z) + O(\varepsilon^2) \ , \end{aligned}$$

and thus

$$\tilde{H}(\Lambda v_1, \Lambda z, \ldots, \Lambda z) = \Lambda \tilde{H}(v_1, z, \ldots, z) \ ; \tag{1.4}$$

in particular (1.3) is a consequence of (1.1).

Now put $z = z_1 + \varepsilon v_2$ in (1.4). We get

$$\begin{aligned} &\tilde{H}(\Lambda v_1, \Lambda z_1, \ldots, \Lambda z_1) + \varepsilon(d-1)\tilde{H}(\Lambda v_1, \Lambda v_2, \Lambda z_1, \ldots, \Lambda z_1) + O(\varepsilon^2) \\ &\qquad = \tilde{H}\big(\Lambda v_1, \Lambda(z_1 + \varepsilon v_2), \ldots, \Lambda(z_1 + \varepsilon v_2)\big) \\ &\qquad = \Lambda \tilde{H}(v_1, z_1 + \varepsilon v_2, \ldots, z_1 + \varepsilon v_2) \\ &\qquad = \Lambda \tilde{H}(v_1, z_1, \ldots, z_1) + \varepsilon(d-1)\Lambda \tilde{H}(v_1, v_2, z, \ldots, z) + O(\varepsilon^2) \ , \end{aligned}$$

and hence

$$\tilde{H}(\Lambda v_1, \Lambda v_2, \Lambda z_1, \ldots, \Lambda z_1) = \Lambda \tilde{H}(v_1, v_2, z_1, \ldots, z_1)$$

for all v_1, v_2, $z_1 \in \mathbb{C}^n$. Proceeding in this way we get (1.2). $\square$

We now introduce the operator needed for the second order normalization.

Definition 1.4: Given $P \in \mathcal{H}^\mu$ and $\Lambda \in M_{n,n}(\mathbb{C})$, let $L_{P,\Lambda}\colon \mathcal{H}^d \to \mathcal{H}^{d+\mu-1}$ be given by

$$L_{P,\Lambda}(H)(z) = d\,\tilde{H}\big(P(z), \Lambda z, \dots, \Lambda z\big) - \mu \tilde{P}\big(H(z), z, \dots, z\big)\ .$$

Remark 1.2: Equation (1.1) implies that

$$d\,\tilde{H}\big(P(z), \Lambda z, \dots, \Lambda z\big) = (\operatorname{Jac} H)(\Lambda z) \cdot P(z)\ .$$

Therefore

$$L_{P,\Lambda}(H) = \big((\operatorname{Jac} H) \circ \Lambda\big) \cdot P - (\operatorname{Jac} P) \cdot H\ ;$$

In the notations of [WZP2] we have $L_{P,\Lambda}(H) = [H, P]$, and $L_{P,\Lambda}|_{\mathcal{H}^d_\Lambda} = \mathcal{T}_d[P]$ when $P \in \mathcal{H}^\mu_\Lambda$.

Using multilinear maps it is easy to prove the following useful fact (cp. [WZP2, Lemma 2.1]):

Lemma 1.3: *Take $\Lambda \in M_{n,n}(\mathbb{C})$ and $P \in \mathcal{H}^\mu_\Lambda$. Then $L_{P,\Lambda}(\mathcal{H}^d_\Lambda) \subseteq \mathcal{H}^{d+\mu-1}_\Lambda$ for all $d \geq 2$.*

Proof: Using Lemma 1.2 and the definition of $L_{P,\Lambda}$, if $H \in \mathcal{H}^d_\Lambda$ we get

$$\begin{aligned} L_{P,\Lambda}(H)(\Lambda z) &= d\,\tilde{H}\big(P(\Lambda z), \Lambda^2 z, \dots, \Lambda^2 z\big) - \mu \tilde{P}\big(H(\Lambda z), \Lambda z, \dots, \Lambda z\big) \\ &= d\,\tilde{H}\big(\Lambda P(z), \Lambda^2 z, \dots, \Lambda^2 z\big) - \mu \tilde{P}\big(\Lambda H(z), \Lambda z, \dots, \Lambda z\big) \\ &= d\,\Lambda \tilde{H}\big(P(z), \Lambda z, \dots, \Lambda z\big) - \mu \Lambda \tilde{P}\big(H(z), z, \dots, z\big) \\ &= \Lambda L_{P,\Lambda}(H)(z)\ . \end{aligned}$$

□

To state and prove the main technical result of this section we fix a few more notations.

Definition 1.5: We shall denote by $\widehat{\mathcal{O}}^n = \prod_{d \geq 1} \mathcal{H}^d$ the space of n-tuples of formal power series with vanishing constant term. Furthermore, given $\Lambda \in M_{n,n}(\mathbb{C})$ we shall denote by $\widehat{\mathcal{O}}^n_\Lambda$ the subset of $F \in \widehat{\mathcal{O}}^n$ with $dF_O = \Lambda$. Every $F \in \widehat{\mathcal{O}}^n$ can be written in a unique way as a formal sum

$$F = \sum_{d \geq 1} F_d \tag{1.5}$$

with $F_d \in \mathcal{H}^d$; (1.5) is the *homogeneous expansion* of F, and F_d is the *d-homogeneous term* of F. We shall often write $\{F\}_d$ for F_d. In particular, if $F \in \widehat{\mathcal{O}}^n_\Lambda$ then $\{F\}_1 = \Lambda$.

The homogeneous terms behave in a predictable way with respect to composition and inverse: indeed it is easy to see that if $F = \sum_{d \geq 1} F_d$ and $G = \sum_{d \geq 1} G_d$ are two elements of $\widehat{\mathcal{O}}^n$ then

$$\{F \circ G\}_d = \sum_{\substack{1 \leq r \leq d \\ d_1 + \cdots + d_r = d}} \tilde{F}_r(G_{d_1}, \dots, G_{d_r}) \tag{1.6}$$

for all $d \geq 1$; and that if $\Phi = I + \sum_{d \geq 2} H_d$ belongs to $\widehat{\mathcal{O}}^n_I$ then the homogeneous expansion of the inverse transformation $\Phi^{-1} = I + \sum_{d \geq 2} K_d$ is given by

$$K_d = -H_d - \sum_{\substack{2 \leq r \leq d-1 \\ d_1 + \cdots + d_r = d}} \tilde{K}_r(H_{d_1}, \dots, H_{d_r}) \tag{1.7}$$

for all $d \geq 2$. In particular we have

Lemma 1.4: *Let $\Phi = I + \sum\limits_{d\ge 2} H_d \in \widehat{\mathcal{O}}^n_I$, and let $\Phi^{-1} = I + \sum\limits_{d\ge 2} K_d$ be the homogeneous expansion of the inverse. Then if $H_2, \ldots, H_d$ are Λ-resonant for some $\Lambda \in M_{n,n}(\mathbb{C})$ and $d \ge 2$ then also K_d is.*

Proof: We argue by induction. Assume that $H_2, \ldots, H_d$ are Λ-resonant. If $d = 2$ then $K_2 = -H_2$ and thus K_2 is clearly Λ-resonant. Assume the assertion true for $d-1$; in particular, $K_2, \ldots, K_{d-1}$ are Λ-resonant. Then

$$\begin{aligned} K_d \circ \Lambda &= -H_d \circ \Lambda - \sum_{\substack{2\le r\le d-1\\ d_1+\cdots+d_r=d}} \tilde{K}_r(H_{d_1}\circ\Lambda, \ldots, H_{d_r}\circ\Lambda)\\ &= \Lambda H_d - \sum_{\substack{2\le r\le d-1\\ d_1+\cdots+d_r=d}} \tilde{K}_r(\Lambda H_{d_1}, \ldots, \Lambda H_{d_r}) = \Lambda K_d \end{aligned}$$

because $K_2, \ldots, K_{d-1}$ are Λ-resonant (and we are using Lemma 1.2). □

Definition 1.6: Given $\Lambda \in M_{n,n}(\mathbb{C})$, we shall say that $F \in \widehat{\mathcal{O}}^n$ is *Λ-resonant* if $F\circ\Lambda = \Lambda F$. Clearly, F is Λ-resonant if and only if $\{F\}_d \in \mathcal{H}^d_\Lambda$ for all $d \in \mathbb{N}$.

The main technical result of this section is the following analogue of [WZP2, Theorem 2.4]:

Theorem 1.5: *Given $F \in \widehat{\mathcal{O}}^n_O$, let $F = \Lambda + \sum\limits_{d\ge\mu} F_d$ be its homogeneous expansion, with $F_\mu \ne O$. Then for every $\Phi \in \widehat{\mathcal{O}}^n_I$ with homogeneous expansion $\Phi = I + \sum\limits_{d\ge 2} H_d$ and every $\nu \ge 2$ we have*

$$\{\Phi^{-1}\circ F\circ\Phi\}_\nu = F_\nu - L_\Lambda(H_\nu) - L_{F_\mu,\Lambda}(H_{\nu-\mu+1}) + Q_\nu + R_\nu\ , \tag{1.8}$$

where Q_ν depends only on Λ and on H_γ with $\gamma < \nu$, while R_ν depends only on F_ρ with $\rho < \nu$ and on H_γ with $\gamma < \nu - \mu + 1$, and we put $L_{F_\mu,\Lambda}(H_1) = O$. Furthermore, we have:
- (i) *if $H_2, \ldots, H_{\nu-1} \in \mathcal{H}_\Lambda$ then $Q_\nu = O$; in particular, if Φ is Λ-resonant then $L_\Lambda(H_\nu) = Q_\nu = O$ for all $\nu \ge 2$;*
- (ii) *if Φ is Λ-resonant then $\{\Phi^{-1}\circ F\circ\Phi\}_\nu = O$ for $2 \le \nu < \mu$, $\{\Phi^{-1}\circ F\circ\Phi\}_\mu = F_\mu$, and*
$$\{\Phi^{-1}\circ F\circ\Phi\}_{\mu+1} = F_{\mu+1} - L_{F_\mu,\Lambda}(H_2)\ ;$$
- (iii) *if $F = \Lambda$ then $R_\nu = O$ for all $\nu \ge 2$;*
- (iv) *if $F_2, \ldots, F_{\nu-1}$ and $H_2, \ldots, H_{\nu-\mu}$ are Λ-resonant then R_ν is Λ-resonant.*

Proof: Using twice (1.6) we get

$$\begin{aligned} \{\Phi^{-1}\circ F\circ\Phi\}_\nu &= \sum_{\substack{1\le s\le\nu\\ \nu_1+\cdots+\nu_s=\nu}} \tilde{K}_s(\{F\circ\Phi\}_{\nu_1}, \ldots, \{F\circ\Phi\}_{\nu_s})\\ &= \sum_{\substack{1\le s\le\nu\\ \nu_1+\cdots+\nu_s=\nu}} \sum_{\substack{1\le r_1\le\nu_1\\ d_{11}+\cdots+d_{1r_1}=\nu_1}} \cdots \sum_{\substack{1\le r_s\le\nu_s\\ d_{s1}+\cdots+d_{sr_s}=\nu_s}} \tilde{K}_s\bigl(\tilde{F}_{r_1}(H_{d_{11}},\ldots,H_{d_{1r_1}}), \ldots, \tilde{F}_{r_s}(H_{d_{s1}},\ldots,H_{d_{sr_s}})\bigr)\\ &= T_\nu + S_1(\nu) + \sum_{s\ge 2} S_s(\nu)\ , \end{aligned}$$

where $\Phi^{-1} = I + \sum\limits_{d\ge 2} K_d$ is the homogeneous expansion of Φ^{-1}, and:

$$\text{(1)} \qquad T_\nu = \sum_{\substack{1\le s\le\nu\\ \nu_1+\cdots+\nu_s=\nu}} \tilde{K}_s(\Lambda H_{\nu_1}, \ldots, \Lambda H_{\nu_s})$$

is obtained considering only the terms with $r_1 = \ldots = r_s = 1$;

(2)
$$S_1(\nu) = \sum_{\substack{\mu \le r \le \nu \\ d_1+\cdots+d_r=\nu}} \tilde{F}_r(H_{d_1}, \ldots, H_{d_r})$$

contains the terms with $s = 1$ and $r_1 > 1$; and

(3)
$$S_s(\nu) = \sum_{\nu_1+\cdots+\nu_s=\nu} \sum_{\substack{1\le r_1 \le \nu_1 \\ \vdots \\ 1 \le r_s \le \nu_s \\ \max\{r_1,\ldots,r_s\}\ge\mu}} \sum_{\substack{d_{11}+\cdots+d_{1r_1}=\nu_1 \\ \vdots \\ d_{s1}+\cdots+d_{sr_s}=\nu_s}} \tilde{K}_s\big(\tilde{F}_{r_1}(H_{d_{11}}, \ldots, H_{d_{1r_1}}), \ldots, \tilde{F}_{r_s}(H_{d_{s1}}, \ldots, H_{d_{sr_s}})\big)$$

contains the terms with fixed $s \ge 2$ and at least one r_j greater than 1 (and thus greater than or equal to μ, because $F_2 = \ldots = F_{\mu-1} = O$ by assumption).

Let us first study T_ν. The summand corresponding to $s = 1$ is ΛH_ν; the summand corresponding to $s = \nu$ is $K_\nu \circ \Lambda$; therefore

$$T_\nu = \Lambda H_\nu + K_\nu \circ \Lambda + \sum_{\substack{2\le s\le \nu-1 \\ \nu_1+\cdots+\nu_s=\nu}} \tilde{K}_s(\Lambda H_{\nu_1}, \ldots, \Lambda H_{\nu_s}) = -L_\Lambda(H_\nu) + Q_\nu \ ,$$

where, using (1.7) to express K_ν,

$$Q_\nu = \sum_{\substack{2\le s\le \nu-1 \\ \nu_1+\cdots+\nu_s=\nu}} \Big[\tilde{K}_s(\Lambda H_{\nu_1}, \ldots, \Lambda H_{\nu_s}) - \tilde{K}_s(H_{\nu_1}\circ\Lambda, \ldots H_{\nu_s}\circ\Lambda)\Big]$$

depends only on Λ and H_γ with $\gamma < \nu$ because $2 \le s \le \nu - 1$ in the sum. In particular, if $H_1, \ldots, H_{\nu-1} \in \mathcal{H}_\Lambda$ then $Q_\nu = O$, and (i) is proved.

Now let us study $S_1(\nu)$. First of all, we clearly have $S_1(\nu) = O$ for $2 \le \nu < \mu$, and $S_1(\mu) = F_\mu$. When $\nu > \mu$ we can write

$$\begin{aligned} S_1(\nu) &= F_\nu + \sum_{\substack{\mu\le r\le \nu-1 \\ d_1+\cdots+d_r=\nu}} \tilde{F}_r(H_{d_1}, \ldots, H_{d_r}) \\ &= F_\nu + \mu\tilde{F}_\mu(H_{\nu-\mu+1}, I, \ldots, I) + \sum_{\substack{d_1+\cdots+d_\mu=\nu \\ 1<\max\{d_j\}<\nu-\mu+1}} \tilde{F}_\mu(H_{d_1}, \ldots, H_{d_\mu}) + \sum_{\substack{\mu+1\le r\le \nu-1 \\ d_1+\cdots+d_r=\nu}} \tilde{F}_r(H_{d_1}, \ldots, H_{d_r}) \ . \end{aligned}$$

in particular, $S_1(\mu + 1) = F_{\mu+1} + \mu\tilde{F}_\mu(H_2, I, \ldots, I)$. Notice that the two remaining sums depend only on F_ρ with $\rho < \nu$ and on H_γ with $\gamma < \nu - \mu + 1$ (in the first sum is clear; for the second one, if $d_j \ge \nu - \mu + 1$ for some j we then would have $d_1 + \cdots + d_r \ge \nu - \mu + 1 + r - 1 \ge \nu + 1$, impossible). Summing up we have

$$S_1(\nu) = \begin{cases} O & \text{for } 2 \le \nu < \mu, \\ F_\mu & \text{for } \nu = \mu, \\ F_{\mu+1} + \mu\tilde{F}_\mu(H_2, I, \ldots, I) & \text{for } \nu = \mu+1, \\ F_\nu + \mu\tilde{F}_\mu(H_{\nu-\mu+1}, I, \ldots, I) + R^1_\nu & \text{for } \nu > \mu+1, \end{cases}$$

where

$$R^1_\nu = \sum_{\substack{d_1+\cdots+d_\mu=\nu \\ 1<\max\{d_j\}<\nu-\mu+1}} \tilde{F}_\mu(H_{d_1}, \ldots, H_{d_\mu}) + \sum_{\substack{\mu+1\le r\le \nu-1 \\ d_1+\cdots+d_r=\nu}} \tilde{F}_r(H_{d_1}, \ldots, H_{d_r})$$

depends only on F_ρ with $\rho < \nu$ and on H_γ with $\gamma < \nu - \mu + 1$.

Let us now discuss $S_s(\nu)$ for $s \geq 2$. First of all, the condition $\max\{r_1, \dots, r_s\} \geq \mu$ implies

$$\mu + s - 1 \leq r_1 + \cdots + r_s \leq \nu_1 + \cdots + \nu_s = \nu ,$$

that is $s \leq \nu - \mu + 1$. In particular, $S_s(\nu) = O$ if $\nu \leq \mu$ or if $s > \nu - \mu + 1$. Moreover, if we had $d_{ij} \geq \nu - \mu + 1$ for some $1 \leq i \leq s$ and $1 \leq j \leq r_s$ we would get

$$\nu = d_{11} + \cdots + d_{sr_s} \geq \nu - \mu + 1 + r_1 + \cdots + r_s - 1 \geq \nu - \mu + 1 + \mu + s - 1 - 1 = \nu + s - 1 > \nu ,$$

impossible. This means that $S_s(\nu)$ depends only on F_ρ with $\rho < \nu$ for all s, on H_γ with $\gamma < \nu - \mu + 1$ when $s < \nu - \mu + 1$, and that $S_{\nu-\mu+1}(\nu)$ depends on $H_{\nu-\mu+1}$ just because it contains $\tilde{K}_{\nu-\mu+1}$. Furthermore, the conditions $\max\{r_1, \dots, r_{\nu-\mu+1}\} \geq \mu$ and $\nu_1 + \ldots + \nu_{\nu-\mu+1} = \nu$ imply that

$$S_{\nu-\mu+1}(\nu) = (\nu - \mu + 1)\tilde{K}_{\nu-\mu+1}(F_\mu, \Lambda, \dots, \Lambda) = -(\nu - \mu + 1)\tilde{H}_{\nu-\mu+1}(F_\mu, \Lambda, \dots, \Lambda) + R^2_\nu ,$$

where (using Lemmas 1.1 and (1.7))

$$R^2_\nu = \sum_{\substack{2 \leq r \leq \nu-\mu \\ d_1 + \cdots + d_r = \nu - \mu + 1}} \sum_{j=1}^{r} d_j \tilde{K}_r\bigl(H_{d_1} \circ \Lambda, \dots, \tilde{H}_{d_j}(F_\mu, \Lambda, \dots, \Lambda), \dots, H_{d_r} \circ \Lambda\bigr)$$

depends only on Λ, F_μ and H_γ with $\gamma < \nu - \mu + 1$.

Putting everything together, we have

$$\begin{aligned}
\{\Phi^{-1} \circ F \circ \Phi\}_\nu &= T_\nu + S_1(\nu) + \sum_{s=2}^{\nu-\mu+1} S_s(\nu) \\
&= F_\nu - L_\Lambda(H_\nu) + Q_\nu + \begin{cases} O & \text{if } 2 \leq \nu \leq \mu, \\ -L_{F_\mu,\Lambda}(H_2) & \text{if } \nu = \mu + 1, \\ -L_{F_\mu,\Lambda}(H_{\nu-\mu+1}) + R_\nu & \text{if } \nu > \mu + 1, \end{cases}
\end{aligned}$$

where

$$R_\nu = R^1_\nu + R^2_\nu + \sum_{s=2}^{\nu-\mu} S_s(\nu)$$

depends only on F_ρ with $\rho < \mu$ and on H_γ with $\gamma < \nu - \mu + 1$. In particular, if $F = \Lambda$ then we have $S_s(\nu) = O$ for all $s \geq 1$ and hence $R_\nu = O$ for all $\nu \geq 2$.

In this way we have proved (1.8) and parts (i), (ii) and (iii). Concerning (iv), it suffices to notice that if $F_2, \dots, F_{\nu-1}$ and $H_2, \dots, H_{\nu-\mu+1}$ are Λ-resonant, then also R^1_ν, $S_2(\nu), \dots, S_{\nu-\mu}(\nu)$ and R^2_ν (by Lemmas 1.2 and 1.4) are Λ-resonant. □

Remark 1.3: In [WZP2] the remainder term R_ν is expressed by using combinations of higher order derivatives instead of combinations of multilinear maps.

We can now introduce the second order normal forms, using the Fischer Hermitian product to provide suitable complementary spaces.

Definition 1.7: The *Fischer Hermitian product* on $\mathcal{H}$ is defined by

$$\langle z_1^{p_1}\cdots z_n^{p_n}e_h, z_1^{q_1}\cdots z_n^{q_n}e_k\rangle = \begin{cases} 0 & \text{if } h\neq k \text{ or } p_j\neq q_j \text{ for some } j;\\ \dfrac{p_1!\cdots p_n!}{(p_1+\cdots+p_n)!} & \text{if } h=k \text{ and } p_j=q_j \text{ for all } j.\end{cases}$$

Definition 1.8: Given $\Lambda\in M_{n,n}(\mathbb{C})$, we shall say that a $G\in\widehat{\mathcal{O}}_\Lambda^n$ is *in second order normal form* if $G=\Lambda$ or the homogeneous expansion $G=\Lambda+\sum\limits_{d\geq\mu}G_d$ of G satisfies the following conditions:

(a) $G_\mu\neq O$;
(b) $G_d\in\mathcal{H}^d\cap(\operatorname{Im}L_{G_\mu,\Lambda})^\perp$ for all $d>\mu$ (where we are using Fischer Hermitian product).

Given $F\in\widehat{\mathcal{O}}_\Lambda^n$, we shall say that $G\in\widehat{\mathcal{O}}_\Lambda^n$ is *a second order normal form* of F if G is in second order normal form and $G=\Phi^{-1}\circ F\circ\Phi$ for some $\Phi\in\widehat{\mathcal{O}}_I^n$.

We can now prove the existence of second order normal forms:

Theorem 1.6: *Let $\Lambda\in M_{n,n}(\mathbb{C})$ be given. Then each $F\in\widehat{\mathcal{O}}_\Lambda^n$ admits a second order normal form. More precisely, if $F=\Lambda+\sum\limits_{d\geq\mu}F_d$ is in Poincaré-Dulac normal form (and $F\not\equiv\Lambda$) then there exists a unique Λ-resonant $\Phi=I+\sum\limits_{d\geq 2}H_d\in\widehat{\mathcal{O}}_I^n$ such that $H_d\in(\operatorname{Ker}L_{F_\mu,\Lambda})^\perp$ for all $d\geq 2$ and $G=\Phi^{-1}\circ F\circ\Phi$ is in second order normal form. Furthermore, if Λ is diagonal we also have $G_d\in\mathcal{H}_\Lambda^d$ for all $d\geq\mu$.*

Proof: By the classical theory we can assume that F is in Poincaré-Dulac normal form. If $F\equiv\Lambda$ we are done; assume then that $F\not\equiv\Lambda$.

First of all, by Theorem 1.5 if Φ is Λ-resonant we have $\{\Phi^{-1}\circ F\circ\Phi\}_d=F_d$ for all $d\leq\mu$. Now consider the splittings

$$\mathcal{H}^d=\operatorname{Im}L_{F_\mu,\Lambda}|_{\mathcal{H}_\Lambda^{d-\mu+1}}\bigoplus(\operatorname{Im}L_{F_\mu,\Lambda}|_{\mathcal{H}_\Lambda^{d-\mu+1}})^\perp$$

and

$$\mathcal{H}_\Lambda^{d-\mu+1}=\operatorname{Ker}L_{F_\mu,\Lambda}|_{\mathcal{H}_\Lambda^{d-\mu+1}}\bigoplus(\operatorname{Ker}L_{F_\mu,\Lambda}|_{\mathcal{H}_\Lambda^{d-\mu+1}})^\perp\ .$$

If $d=\mu+1$ we can find a unique $G_{\mu+1}\in(\operatorname{Im}L_{F_\mu,\Lambda})^\perp\cap\mathcal{H}^{\mu+1}$ and a unique $H_2\in(\operatorname{Ker}L_{F_\mu,\Lambda})^\perp\cap\mathcal{H}_\Lambda^2$ such that $F_{\mu+1}=G_{\mu+1}+L_{F_\mu,\Lambda}(H_2)$. Then Theorem 1.5 yields

$$\{\Phi^{-1}\circ F\circ\Phi\}_{\mu+1}=F_{\mu+1}-L_{F_\mu,\Lambda}(\{\Phi\}_2)=G_{\mu+1}+L_{F_\mu,\Lambda}(H_2)-L_{F_\mu,\Lambda}(\{\Phi\}_2)\ ;$$

so to get $\{\Phi^{-1}\circ F\circ\Phi\}_{\mu+1}\in(\operatorname{Im}L_{F_\mu,\Lambda})^\perp\cap\mathcal{H}^{\mu+1}$ with $\{\Phi\}_2\in(\operatorname{Ker}L_{F_\mu,\Lambda})^\perp\cap\mathcal{H}_\Lambda^2$ we must necessarily take $\{\Phi\}_2=H_2$.

Assume, by induction, that we have uniquely determined $H_2,\ldots,H_{d-\mu}\in(\operatorname{Im}L_{F_\mu,\Lambda})^\perp\cap\mathcal{H}_\Lambda$, and thus $R_d\in\mathcal{H}^d$ in (1.8). Hence there is a unique $G_d\in(\operatorname{Im}L_{F_\mu,\Lambda})^\perp\cap\mathcal{H}^d$ and a unique $H_{d-\mu+1}\in(\operatorname{Ker}L_{F_\mu,\Lambda})^\perp\cap\mathcal{H}_\Lambda^{d-\mu+1}$ such that $F_d+R_d=G_d+L_{F_\mu,\Lambda}(H_{d-\mu+1})$. Thus to get $\{\Phi^{-1}\circ F\circ\Phi\}_d\in(\operatorname{Im}L_{F_\mu,\Lambda})^\perp\cap\mathcal{H}^d$ with $\{\Phi\}_{d-\mu+1}\in(\operatorname{Ker}L_{F_\mu,\Lambda})^\perp\cap\mathcal{H}_\Lambda^{d-\mu+1}$ the only possible choice is $\{\Phi\}_{d-\mu+1}=H_{d-\mu+1}$, and thus $\{\Phi^{-1}\circ F\circ\Phi\}_d=G_d$.

Finally, if Λ is diagonal then $F_d\in\mathcal{H}_\Lambda^d$ for all $d\geq\mu$. Furthermore, Lemma 1.3 yields $\operatorname{Im}L_{F_\mu,\Lambda}|_{\mathcal{H}_\Lambda^{d-\mu+1}}\subseteq\mathcal{H}_\Lambda^d$ for all $d\geq\mu$; recalling Theorem 1.5.(vi) we then see can we can always find $G_d\in\mathcal{H}_\Lambda^d$, and we are done. □

The definition and construction of k-th order normal forms is similar; the idea is to extract from the remainder term R_ν the pieces depending on H_γ with γ varying in a suitable range, and use them to build operators generalizing L_Λ and $L_{P,\Lambda}$. We refer to [WZP2] for details; for our needs it suffices to recall that given $F = \Lambda + \sum_{d\geq 2} F_d \in \widehat{\mathcal{O}}^n_\Lambda$ [WPZ2] introduces a sequence of (not necessarily linear) operators $\mathcal{L}^{(d)}[\Lambda, F_2, \dots, F_d]\colon \operatorname{Ker}\mathcal{L}^{(d-1)} \times \mathcal{H}^{d+1} \to \mathcal{H}^{d+1}$ for $d \geq 1$, with $\mathcal{L}^{(1)}[\Lambda](H_2) = L_\Lambda(H_2)$ and $\mathcal{L}^{(2)}[\Lambda, F_2](H_2, H_3) = L_\Lambda(H_3) + L_{F_2,\Lambda}(H_2)$, and gives the following definition:

Definition 1.9: We shall say that $G = \Lambda + \sum\limits_{d\geq 2} G_d \in \widehat{\mathcal{O}}^n_\Lambda$ is in *infinite order normal form* if $G_d \in W_d^\perp$ for all $d \geq 2$, where W_d is a vector subspace of maximal dimension contained in the image of $\mathcal{L}^{(d-1)}[\Lambda, G_2, \dots, G_{d-1}]$. We shall also say that G is an *infinite order normal form of* $F \in \widehat{\mathcal{O}}^n_\Lambda$ if it is in infinite order normal form and it is formally conjugated to F.

We end this section quoting a result from [WZP2] giving a condition ensuring that a second order normal form is actually an infinite order normal form:

Proposition 1.7: ([WZP2, Theorem 4.9]) *Let $\Lambda \in M_{n,n}(\mathbb{C})$ be diagonal, and $F = \Lambda + \sum\limits_{d\geq 2} F_d \in \widehat{\mathcal{O}}^n_\Lambda$ with $F_2 \neq O$ and Λ-resonant. Assume that $\operatorname{Ker} L_{F_2,\Lambda}|_{\mathcal{H}^d_\Lambda} = \{O\}$ for all $d \geq 2$. Then the second order normal form of F is the unique infinite order normal form of F.*

2. Superattracting germs

In this section we shall completely describe the second order normal forms obtained when $n = \mu = 2$ and $\Lambda = O$, that is for 2-dimensional superattracting germs with non-vanishing quadratic term. Except in four degenerate instances, the second order normal forms will be infinite order normal forms, and will be expressed just in terms of two power series of *one* variable, thus giving a drastic simplification of the germs.

In [A1] we showed that, up to a linear change of variable, we can assume that the quadratic term F_2 is of one (and only one) of the following forms:

(∞) $F_2(z,w) = (z^2, zw)$;
(1_{00}) $F_2(z,w) = (0, -z^2)$;
(1_{10}) $F_2(z,w) = \big(-z^2, -(z^2 + zw)\big)$;
(1_{11}) $F_2(z,w) = \big(-zw, -(z^2 + w^2)\big)$;
(2_{001}) $F_2(z,w) = (0, zw)$;
(2_{011}) $F_2(z,w) = (zw, zw + w^2)$;
($2_{10\rho}$) $F_2(z,w) = \big(-\rho z^2, (1-\rho)zw\big)$, with $\rho \neq 0$;
($2_{11\rho}$) $F_2(z,w) = \big(\rho z^2 + zw, (1+\rho)zw + w^2\big)$, with $\rho \neq 0$;
(3_{100}) $F_2(z,w) = (z^2 - zw, 0)$;
($3_{\rho 10}$) $F_2(z,w) = \big(\rho(-z^2 + zw), (1-\rho)(zw - w^2)\big)$, with $\rho \neq 0$, 1;
($3_{\rho\tau 1}$) $F_2(z,w) = \big(-\rho z^2 + (1-\tau)zw, (1-\rho)zw - \tau w^2\big)$, with ρ, $\tau \neq 0$ and $\rho + \tau \neq 1$

(where the symbols refer to the number of characteristic directions and to their indeces; see also [AT2]).

We shall use the standard basis $\{u_{d,j}, v_{d,j}\}_{j=0,\dots,d}$ of $\mathcal{H}^d$, where

$$u_{d,j} = (z^j w^{d-j}, 0) \qquad \text{and} \qquad v_{d,j} = (0, z^j w^{d-j}) \ ,$$

and we shall endow $\mathcal{H}^d$ with Fischer Hermitian product, so that $\{u_{d,j}, v_{d,j}\}_{j=0,\dots,d}$ is an orthogonal basis and

$$\|u_{d,j}\|^2 = \|v_{d,j}\|^2 = \binom{d}{j}^{-1} .$$

When $\Lambda = O$, we have $\mathcal{H}_\Lambda = \mathcal{H}$, and the operator $L = L_{F_2,\Lambda}\colon \mathcal{H}^d \to \mathcal{H}^{d+1}$ is given by

$$L(H) = -\operatorname{Jac}(F_2) \cdot H \ .$$

To apply Proposition 1.7, we need to know when $\operatorname{Ker} L|_{\mathcal{H}^d} = \{O\}$. Since

$$\dim \operatorname{Ker} L|_{\mathcal{H}^d} + \dim \operatorname{Im} L|_{\mathcal{H}^d} = \dim \mathcal{H}^d = \dim \mathcal{H}^{d+1} - 2 = \dim \operatorname{Im} L|_{\mathcal{H}^d} + \dim(\operatorname{Im} L|_{\mathcal{H}^d})^\perp - 2 \ ,$$

we find that

$$\operatorname{Ker} L|_{\mathcal{H}^d} = \{O\} \quad \text{if and only if} \quad \dim(\operatorname{Im} L|_{\mathcal{H}^d})^\perp = 2 \ . \tag{2.1}$$

We shall now study separately each case.

• *Case* (∞).
In this case we have

$$L(u_{d,j}) = -2u_{d+1,j+1} - v_{d+1,j} \qquad \text{and} \qquad L(v_{d,j}) = -v_{d+1,j+1}$$

for all $d \geq 2$ and $j = 0, \ldots, d$. Therefore

$$\operatorname{Im} L|_{\mathcal{H}^d} = \operatorname{Span}\left(u_{d+1,2}, \ldots, u_{d+1,d+1}, 2u_{d+1,1} + v_{d+1,0}, v_{d+1,1}, \ldots, v_{d+1,d+1}\right) \ ,$$

and thus

$$(\operatorname{Im} L|_{\mathcal{H}^d})^\perp = \operatorname{Span}\left(u_{d+1,0}, (d+1)u_{d+1,1} - 2v_{d+1,0}\right) \ .$$

In particular, thanks to (2.1) and Proposition 1.7, a second order normal form is automatically an infinite order normal form.

It then follows that every formal power series of the form

$$F(z,w) = (z^2 + O_3, zw + O_3)$$

(where O_3 denotes a remainder term of order at least 3) has a unique infinite order normal form

$$G(z,w) = \bigl(z^2 + \varphi(w) + z\psi'(w), zw - 2\psi(w)\bigr)$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3. Notice that (here and in later formulas) the appearance of the derivative (which simplifies the expression of the normal form) is due to the fact we are using the Fischer Hermitian product; using another Hermitian product might lead to more complicated normal forms.

• *Case* (1_{00}).
In this case we have

$$L(u_{d,j}) = 2v_{d+1,j+1} \qquad \text{and} \qquad L(v_{d,j}) = 0$$

for all $d \geq 2$ and $j = 0, \ldots, d$. Therefore

$$\operatorname{Im} L|_{\mathcal{H}^d} = \operatorname{Span}\left(v_{d+1,1}, \ldots, v_{d+1,d+1}\right) \ ,$$

and thus

$$(\operatorname{Im} L|_{\mathcal{H}^d})^\perp = \operatorname{Span}\left(u_{d+1,0}, \ldots, u_{d+1,d+1}, v_{d+1,0}\right) \ .$$

This a degenerate case, where we cannot use Proposition 1.7. Anyway, Theorem 1.6 still apply, and it follows that every formal power series of the form

$$F(z,w) = (O_3, -z^2 + O_3)$$

has a second order normal form

$$G(z,w) = \bigl(\Phi(z,w), -z^2 + \psi(w)\bigr)$$

where $\psi \in \mathbb{C}[\![\zeta]\!]$ and $\Phi \in \mathbb{C}[\![z,w]\!]$ are power series of order at least 3.

• *Case* (1_{10}).
In this case we have

$$L(u_{d,j}) = 2u_{d+1,j+1} + 2v_{d+1,j+1} + v_{d+1,j} \qquad \text{and} \qquad L(v_{d,j}) = v_{d+1,j+1}$$

for all $d \geq 2$ and $j = 0, \ldots, d$. Therefore

$$\operatorname{Im} L|_{\mathcal{H}^d} = \operatorname{Span}\left(2u_{d+1,1} + v_{d+1,0}, u_{d+1,2}, \ldots, u_{d+1,d+1}, v_{d+1,1}, \ldots, v_{d+1,d+1}\right) ,$$

and thus

$$(\operatorname{Im} L|_{\mathcal{H}^d})^\perp = \operatorname{Span}\left(u_{d+1,0}, (d+1)u_{d+1,1} - 2v_{d+1,0}\right) .$$

It then follows that every formal power series of the form

$$F(z,w) = (-z^2 + O_3, -z^2 - zw + O_3)$$

has a unique infinite order normal form

$$G(z,w) = \bigl(-z^2 + \varphi(w) + z\psi'(w), -z^2 - zw - 2\psi(w)\bigr)$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3.

• *Case* (1_{11}).
In this case we have

$$L(u_{d,j}) = u_{d+1,j} + 2v_{d+1,j+1} \qquad \text{and} \qquad L(v_{d,j}) = u_{d+1,j+1} + 2v_{d+1,j}$$

for all $d \geq 2$ and $j = 0, \ldots, d$. It follows that

$$\begin{aligned}\operatorname{Im} L|_{\mathcal{H}^d} = \operatorname{Span}(&u_{d+1,0} - u_{d+1,2}, \ldots, u_{d+1,d-1} - u_{d+1,d+1},\\ &v_{d+1,2} - v_{d+1,0}, \ldots, v_{d+1,d+1} - v_{d+1,d-1}, u_{d+1,0} + 2v_{d+1,1}, u_{d+1,1} + 2v_{d+1,0}) ,\end{aligned}$$

and a few computations yield

$$\begin{aligned}(\operatorname{Im} L|_{\mathcal{H}^d})^\perp &= \operatorname{Span}\left(\sum_{j=0}^{d+1}\binom{d+1}{j}(v_{d+1,j} - 2u_{d+1,j}), \sum_{j=0}^{d+1}(-1)^j\binom{d+1}{j}(v_{d+1,j} + 2u_{d+1,j})\right) .\\ &= \operatorname{Span}\left(\bigl(-2(z+w)^{d+1}, (z+w)^{d+1}\bigr), \bigl(2(w-z)^{d+1}, (w-z)^{d+1}\bigr)\right)\end{aligned}$$

It then follows that every formal germ of the form

$$F(z,w) = (-zw + O_3, -z^2 - w^2 + O_3)$$

has a unique infinite order normal form

$$G(z,w) = \bigl(-zw - 2\varphi(z+w) + 2\psi(w-z), -z^2 - w^2 + \varphi(z+w) + \psi(w-z)\bigr)$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are arbitrary power series of order at least 3. Again, the fact that the normal form is expressed in terms of power series evaluated in $z + w$ and $z - w$ is due to the fact we are using Fischer Hermitian product.

• *Case* (2_{001}).
In this case we have

$$L(u_{d,j}) = -v_{d+1,j} \qquad \text{and} \qquad L(v_{d,j}) = -v_{d+1,j+1}$$

for all $d \geq 2$ and $j = 0, \ldots, d$. It follows that

$$\operatorname{Im} L|_{\mathcal{H}^d} = \operatorname{Span}(v_{d+1,0}, \ldots, v_{d+1,d+1})$$

and hence

$$(\operatorname{Im} L|_{\mathcal{H}^d})^\perp = \operatorname{Span}(u_{d+1,0}, \ldots, u_{d+1,d+1}) \ .$$

We are in a degenerate case; hence every formal germ of the form

$$F(z,w) = (O_3, zw + O_3)$$

has a second order normal form

$$G(z,w) = \big(\Phi(z,w), zw\big)$$

where $\Phi \in \mathbb{C}[\![z,w]\!]$ is a power series of order at least three.

• *Case* (2_{011}).
In this case we have

$$L(u_{d,j}) = -u_{d+1,j} - v_{d+1,j} \qquad \text{and} \qquad L(v_{d,j}) = -u_{d+1,j+1} - 2v_{d+1,j} - v_{d+1,j+1}$$

for all $d \geq 2$ and $j = 0, \ldots, d$. It follows that

$$\begin{aligned} \operatorname{Im} L|_{\mathcal{H}^d} = \operatorname{Span}(&u_{d+1,0}, \ldots, u_{d+1,d-1}, v_{d+1,0}, \ldots, v_{d+1,d-1}, \\ &u_{d+1,d} + v_{d+1,d}, u_{d+1,d+1} + v_{d+1,d+1} + 2v_{d+1,d}) \ , \end{aligned}$$

and hence

$$(\operatorname{Im} L|_{\mathcal{H}^d})^\perp = \operatorname{Span}\left((d+1)u_{d+1,d} - (d+1)v_{d+1,d} + 2v_{d+1,d+1}, u_{d+1,d+1} - v_{d+1,d+1}\right) \ .$$

It then follows that every formal germ of the form

$$F(z,w) = (zw + O_3, zw + w^2 + O_3)$$

has a unique infinite order normal form

$$G(z,w) = \big(zw + w\varphi'(z) + \psi(z), zw + w^2 + 2\varphi(z) - w\varphi'(z) - \psi(z)\big) \ ,$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3.

• *Case* $(2_{10\rho})$.
In this case we have

$$L(u_{d,j}) = 2\rho u_{d+1,j+1} + (\rho - 1)v_{d+1,j} \qquad \text{and} \qquad L(v_{d,j}) = (\rho - 1)v_{d+1,j+1}$$

for all $d \geq 2$ and $j = 0, \ldots, d$. We clearly have two subcases to consider.

If $\rho = 1$ then

$$\operatorname{Im} L|_{\mathcal{H}^d} = \operatorname{Span}(u_{d+1,1}, \ldots, u_{d+1,d+1}) \ ,$$

and hence

$$(\operatorname{Im} L|_{\mathcal{H}^d})^\perp = \operatorname{Span}\left(u_{d+1,0}, v_{d+1,0}, \ldots, v_{d+1,d+1}\right) .$$

We are in the third degenerate case; hence every formal germ of the form

$$F(z,w) = (-z^2 + O_3, O_3)$$

has a second order normal form

$$G(z,w) = \left(-z^2 + \psi(w), \Phi(z,w)\right) ,$$

where $\psi \in \mathbb{C}[\![\zeta]\!]$ and $\Phi \in \mathbb{C}[\![z,w]\!]$ are power series of order at least 3.

If instead $\rho \neq 1$ (recalling that $\rho \neq 0$ too) then

$$\operatorname{Im} L|_{\mathcal{H}^d} = \operatorname{Span}\left(2\rho u_{d+1,1} + (\rho-1)v_{d+1,0}, u_{d+1,2}, \ldots, u_{d+1,d+1}, v_{d+1,1}, \ldots, v_{d+1,d+1}\right) ,$$

and hence

$$(\operatorname{Im} L|_{\mathcal{H}^d})^\perp = \operatorname{Span}\left(u_{d+1,0}, (\rho-1)(d+1)u_{d+1,1} - 2\rho v_{d+1,0}\right) .$$

It then follows that every formal germ of the form

$$F(z,w) = \left(-\rho z^2 + O_3, (1-\rho)zw + O_3\right)$$

with $\rho \neq 0$, 1 has a unique infinite order normal form

$$G(z,w) = \left(-\rho z^2 + (\rho-1)z\varphi'(w) + \psi(w), (1-\rho)zw - 2\rho\varphi(z)\right) ,$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3.

• *Case* $(2_{11\rho})$.
In this case we have

$$\begin{cases} L(u_{d,j}) = -2\rho u_{d+1,j+1} - u_{d+1,j} - (1+\rho)v_{d+1,j} \\ L(v_{d,j}) = -u_{d+1,j+1} - 2v_{d+1,j} - (1+\rho)v_{d+1,j+1} \end{cases} \tag{2.2}$$

for all $d \geq 2$ and $j = 0, \ldots, d$. We clearly have two subcases to consider.

If $\rho = -1$ then

$$\operatorname{Im} L|_{\mathcal{H}^d} = \operatorname{Span}\left(u_{d+1,0} - 2u_{d+1,1}, \ldots, u_{d+1,d} - 2u_{d+1,d+1}, u_{d+1,1} + 2v_{d+1,0}, \ldots, u_{d+1,d} + 2v_{d+1,d}\right) ,$$

and hence

$$\begin{aligned}(\operatorname{Im} L|_{\mathcal{H}^d})^\perp &= \operatorname{Span}\left(\sum_{j=0}^{d+1} \binom{d+1}{j} \frac{1}{2^j}(u_{d+1,j} - \frac{1}{4}v_{d+1,j}), v_{d+1,d+1}\right) \\ &= \operatorname{Span}\left(\left(\left(\tfrac{z}{2} + w\right)^{d+1}, -\frac{1}{4}\left(\tfrac{z}{2} + w\right)^{d+1}\right), (0, z^{d+1})\right) .\end{aligned}$$

It then follows that every formal germ of the form

$$F(z,w) = (-z^2 + zw + O_3, w^2 + O_3)$$

has a unique infinite order normal form

$$G(z,w)=\left(-z^2+zw+\varphi(\tfrac{z}{2}+w),w^2-\frac{1}{4}\varphi(\tfrac{z}{2}+w)+\psi(z)\right)\ ,$$

where φ, $\psi\in\mathbb{C}[\![\zeta]\!]$ are power series of order at least 3.

If instead $\rho\neq-1$ (recalling that $\rho\neq 0$ too) then a basis of $\operatorname{Im}L|_{\mathcal{H}^d}$ is given by the vectors listed in (2.2), and a computation shows that $(\operatorname{Im}L|_{\mathcal{H}^d})^\perp$ is given by homogeneous maps of the form

$$\sum_{j=0}^{d+1}(a_ju_{d+1,j}+b_jv_{d+1,j})$$

where the coefficients a_j, b_j satisfy the following relations:

$$\begin{cases} c_jb_j=-\dfrac{2}{1+\rho}c_{j-1}b_{j-1}-\dfrac{1}{\rho(1+\rho)}c_{j-2}b_{j-2} & \text{for } j=2,\dots,d+1,\\ c_ja_j=\dfrac{1}{\rho}c_{j-2}b_{j-2} & \text{for } j=2,\dots,d+1,\\ a_0=(3\rho-1)b_0+2\dfrac{\rho(1+\rho)}{d+1}b_1\ , & \\ a_1=-2(d+1)b_0-(1+\rho)b_1\ , & \end{cases}$$

where $c_j^{-1}=\binom{d+1}{j}$ and b_0, $b_1\in\mathbb{C}$ are arbitrary. Solving these recurrence equations one gets

$$b_j=\frac{1}{2\sqrt{-\rho}}\binom{d+1}{j}\left[\frac{\rho(1+\rho)}{d+1}(m_\rho^j-n_\rho^j)b_1+\big(\rho(m_\rho^j-n_\rho^j)+\sqrt{-\rho}(m_\rho^j+n_\rho^j)\big)b_0\right]\ ,$$

where $\sqrt{-\rho}$ is any square root of $-\rho$, and

$$m_\rho=\frac{\sqrt{-\rho}-\rho}{\rho(1+\rho)}\ ,\qquad n_\rho=-\frac{\sqrt{-\rho}+\rho}{\rho(1+\rho)}\ .$$

It follows that the unique infinite order normal form of a formal germ of the form

$$F(z,w)=\big(\rho z^2+zw+O_3,(1+\rho)zw+w^2+O_3\big)$$

with $\rho\neq 0$, -1 is

$$\begin{aligned}G(z,w)=\Bigg(&\rho z^2+zw+\frac{1}{\rho}\left[\frac{1-\sqrt{-\rho}}{2m_\rho^2}\varphi(m_\rho z+w)+\frac{1+\sqrt{-\rho}}{2n_\rho^2}\varphi(n_\rho z+w)\right]\\ &+\frac{1+\rho}{2\sqrt{-\rho}}\left(\frac{1}{m_\rho^2}\psi(m_\rho z+w)-\frac{1}{n_\rho^2}\psi(n_\rho z+w)\right),\\ &(1+\rho)zw+w^2+\frac{1-\sqrt{-\rho}}{2}\varphi(m_\rho z+w)+\frac{1+\sqrt{-\rho}}{2}\varphi(n_\rho z+w)\\ &+\frac{\rho(1+\rho)}{2\sqrt{-\rho}}\big(\psi(m_\rho z+w)-\psi(n_\rho z+w)\big)\Bigg)\end{aligned}$$

where φ, $\psi\in\mathbb{C}[\![\zeta]\!]$ are power series of order at least 3.

• *Case* (3_{100}).
In this case we have

$$L(u_{d,j}) = u_{d+1,j} - 2u_{d+1,j+1} \qquad \text{and} \qquad L(v_{d,j}) = u_{d+1,j+1}$$

for all $d \geq 2$ and $j = 0, \ldots, d$. It follows that

$$\operatorname{Im} L|_{\mathcal{H}^d} = \operatorname{Span}(u_{d+1,0}, \ldots, u_{d+1,d+1})$$

and hence

$$(\operatorname{Im} L|_{\mathcal{H}^d})^\perp = \operatorname{Span}(v_{d+1,0}, \ldots, v_{d+1,d+1}) \ .$$

We are in the last degenerate case; hence every formal germ of the form

$$F(z,w) = (z^2 - zw + O_3, O_3)$$

has a second order normal form

$$G(z,w) = \bigl(z^2 - zw, \Phi(z,w)\bigr) \ ,$$

where $\Phi \in \mathbb{C}[\![z,w]\!]$ is a power series of order at least 3.

• *Case* $(3_{\rho 10})$.
In this case we have

$$\begin{cases} L(u_{d,j}) = \rho(2u_{d+1,j+1} - u_{d+1,j}) + (\rho-1)v_{d+1,j} \\ L(v_{d,j}) = -\rho u_{d+1,j+1} + (\rho-1)(v_{d+1,j+1} - 2v_{d+1,j}) \end{cases} \tag{2.3}$$

for all $d \geq 2$ and $j = 0, \ldots, d$. Then a basis of $\operatorname{Im} L|_{\mathcal{H}^d}$ is given by the homogeneous maps listed in (2.3), and a computation shows that $(\operatorname{Im} L|_{\mathcal{H}^d})^\perp$ is given by homogeneous maps of the form

$$\sum_{j=0}^{d+1}(a_j u_{d+1,j} + b_j v_{d+1,j})$$

where the coefficients a_j, b_j satisfy the following relations:

$$\begin{cases} c_{j+1}a_{j+1} = \dfrac{\rho-1}{\rho}(c_{j+1}b_{j+1} - 2c_j b_j) & \text{for } j = 0, \ldots, d, \\ c_{j+1}b_{j+1} = 2c_j b_j - c_{j-1}b_{j-1} & \text{for } j = 1, \ldots, d, \\ c_0 a_0 = 2c_1 a_1 + \dfrac{\rho-1}{\rho} c_0 b_0 \ , \end{cases}$$

where $c_j^{-1} = \binom{d+1}{j}$ and b_0, $b_1 \in \mathbb{C}$ are arbitrary. Solving these recurrence equations we find

$$\begin{cases} b_j = \binom{d+1}{j}\left[\frac{j}{d+1}b_1 - (j-1)b_0\right] & \text{for } j = 0, \ldots, d+1, \\ a_j = \frac{\rho-1}{\rho}\binom{d+1}{j}\left[\frac{2-j}{d+1}b_1 + (j-3)b_0\right] & \text{for } j = 0, \ldots, d+1, \end{cases}$$

where b_0, $b_1 \in \mathbb{C}$ are arbitrary. So every formal germ of the form

$$F(z,w) = \bigl(\rho(-z^2 + zw) + O_3, (1-\rho)(zw - w^2) + O_3\bigr)$$

with $\rho \neq 0$, 1 has a unique infinite order normal form

$$\begin{aligned} G(z,w) = \biggl(&\rho(-z^2 + zw) + z\frac{\partial}{\partial z}\bigl[\varphi(z+w) + \psi(z+w)\bigr] - \varphi(z+w), \\ &(1-\rho)(zw - w^2) + \frac{\rho-1}{\rho}\left(z\frac{\partial}{\partial z}\bigl[\varphi(z+w) - \psi(z+w)\bigr] - 3\varphi(z+w) + 2\psi(z+w)\right)\biggr) \end{aligned}$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3.

• *Case* $(3_{\rho\tau 1})$.
In this case we have

$$L(u_{d,j}) = (\tau-1)u_{d+1,j} + 2\rho u_{d+1,j+1} + (\rho-1)v_{d+1,j}$$

and

$$L(v_{d,j}) = (\tau-1)u_{d+1,j+1} + 2\tau v_{d+1,j} + (\rho-1)v_{d+1,j+1}$$

for all $d \geq 2$ and $j = 0, \ldots, d$. As before, we have a few subcases to consider.

Assume first $\rho = \tau = 1$. Then

$$\operatorname{Im} L|_{\mathcal{H}^d} = \operatorname{Span}(u_{d+1,1}, \ldots, u_{d+1,d+1}, v_{d+1,0}, \ldots, v_{d+1,d}) \ ;$$

hence

$$(\operatorname{Im} L|_{\mathcal{H}^d})^\perp = \operatorname{Span}(u_{d+1,0}, v_{d+1,d+1}) \ ,$$

It then follows that every formal germ of the form

$$F(z,w) = (-z^2 + O_3, -w^2 + O_3)$$

has a unique infinite order normal form

$$G(z,w) = \big(-z^2 + \varphi(w), -w^2 + \psi(z)\big) \ ,$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3.

Assume now $\rho \neq 1$. Then a computation shows that $(\operatorname{Im} L|_{\mathcal{H}^d})^\perp$ is given by homogeneous maps of the form

$$\sum_{j=0}^{d+1} (a_j u_{d+1,j} + b_j v_{d+1,j})$$

where the coefficients a_j, b_j satisfy the following relations:

$$\begin{cases} c_{j+1}a_{j+1} = \dfrac{\tau}{\rho} c_{j-1}b_{j-1} & \text{for } j = 1, \ldots, d, \\ c_{j+1}b_{j+1} = -\dfrac{2\tau}{\rho-1} c_j b_j - \dfrac{\tau(\tau-1)}{\rho(\rho-1)} c_{j-1}b_{j-1} & \text{for } j = 1, \ldots, d, \\ (\tau-1)c_1a_1 + (\rho-1)c_1b_1 + 2\tau c_0 b_0 = 0 \ , \\ (\tau-1)c_0a_0 + (\rho-1)c_0b_0 + 2\rho c_1 a_1 = 0 \ , \end{cases} \tag{2.4}$$

where $c_j^{-1} = \binom{d+1}{j}$ and b_0, $b_1 \in \mathbb{C}$ are arbitrary.

When $\tau = 1$ conditions (2.4) reduce to

$$\begin{cases} c_{j+1}a_{j+1} = \dfrac{1}{\rho} c_{j-1}b_{j-1} & \text{for } j = 1, \ldots, d, \\ c_{j+1}b_{j+1} = -\dfrac{2}{\rho-1} c_j b_j & \text{for } j = 1, \ldots, d, \\ (\rho-1)c_1b_1 + 2c_0b_0 = 0 \ , \\ (\rho-1)c_0b_0 + 2\rho c_1 a_1 = 0 \ , \end{cases}$$

whose solution is

$$\begin{cases} a_j = \binom{d+1}{j}\frac{1}{\rho}\left(\frac{2}{1-\rho}\right)^{j-2} b_0 & \text{for } j=1,\dots,d+1, \\ b_j = \binom{d+1}{j}\left(\frac{2}{1-\rho}\right)^{j} b_0 & \text{for } j=0,\dots,d+1, \end{cases}$$

where a_0, $b_0 \in \mathbb{C}$ are arbitrary. Therefore

$$(\operatorname{Im} L|_{\mathcal{H}^d})^\perp = \operatorname{Span}\left((w^{d+1},0), \left(\frac{(1-\rho)^2}{4\rho}\left(\frac{2}{1-\rho}z+w\right)^{d+1}, \left(\frac{2}{1-\rho}z+w\right)^{d+1}\right)\right) ,$$

and thus every formal germ of the form

$$F(z,w) = \left(-\rho z^2 + O_3, (1-\rho)zw - w^2 + O_3\right)$$

with $\rho \neq 1$ has a unique infinite order normal form

$$G(z,w) = \left(-\rho z^2 + \varphi(w) + \frac{(1-\rho)^2}{4\rho}\psi\left(\frac{2}{1-\rho}z+w\right), (1-\rho)zw - w^2 + \psi\left(\frac{2}{1-\rho}z+w\right)\right) ,$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are arbitrary power series of order at least 3.

The case $\rho = 1$ and $\tau \neq 1$ is treated in the same way; we get that every formal germ of the form

$$F(z,w) = \left(-z^2 + (1-\tau)zw + O_3, -\tau w^2 + O_3\right)$$

with $\tau \neq 1$ has a unique infinite order normal form

$$G(z,w) = \left(-z^2 + (1-\tau)zw + \psi\left(\frac{1-\tau}{2}z+w\right), -\tau w^2 + \varphi(z) + \frac{(1-\tau)^2}{4\tau}\psi\left(\frac{1-\tau}{2}z+w\right)\right) ,$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3.

Finally assume ρ, $\tau \neq 1$ (and $\rho + \tau \neq 1$). Solving the recurrence equations (2.4) we find

$$\begin{aligned} b_j = \frac{1}{2\sqrt{\rho\tau(\rho+\tau-1)}}\binom{d+1}{j}\Bigg[&\frac{\rho(\rho-1)}{d+1}(m^j_{\rho,\tau} - n^j_{\rho,\tau})b_1 \\ &+ \left(\rho\tau(m^j_{\rho,\tau} - n^j_{\rho,\tau}) + \sqrt{\rho\tau(\rho+\tau-1)}(m^j_{\rho,\tau} + n^j_{\rho,\tau})\right)b_0\Bigg] , \end{aligned}$$

for $j = 0,\dots,d+1$, where $\sqrt{\rho\tau(\rho+\tau-1)}$ is any square root of $\rho\tau(\rho+\tau-1)$, and

$$m_{\rho,\tau} = \frac{\sqrt{\rho\tau(\rho+\tau-1)} - \rho\tau}{\rho(\rho-1)} , \qquad n_{\rho,\tau} = -\frac{\sqrt{\rho\tau(\rho+\tau-1)} + \rho\tau}{\rho(\rho-1)} .$$

Moreover, from (2.4) we also get

$$\begin{aligned} a_j = \frac{\tau}{2\rho\sqrt{\rho\tau(\rho+\tau-1)}}\binom{d+1}{j}\Bigg[&\frac{\rho(\rho-1)}{d+1}(m^{j-2}_{\rho,\tau} - n^{j-2}_{\rho,\tau})b_1 \\ &+ \left(\rho\tau(m^{j-2}_{\rho,\tau} - n^{j-2}_{\rho,\tau}) + \sqrt{\rho\tau(\rho+\tau-1)}(m^{j-2}_{\rho,\tau} + n^{j-2}_{\rho,\tau})\right)b_0\Bigg] , \end{aligned}$$

again for $j=0,\dots,d+1$. It follows that the unique infinite order normal form of a formal germ of the form

$$F(z,w)=\bigl(-\rho z^2+(1-\tau)zw+O_3,(1-\rho)zw-\tau w^2+O_3\bigr)$$

with ρ, $\tau\ne 0$, 1 and $\rho+\tau\ne 1$, is

$$\begin{aligned}G(z,w)=\Biggl(-\rho z^2+(1-\tau)zw+\frac{\tau}{\rho}\biggl[&\frac{\sqrt{\rho+\tau-1}+\sqrt{\rho\tau}}{2m_{\rho,\tau}^2}\varphi(m_{\rho,\tau}z+w)\\ &+\frac{\sqrt{\rho+\tau-1}-\sqrt{\rho\tau}}{2n_{\rho,\tau}^2}\varphi(n_{\rho,\tau}z+w)\\ &+\frac{1}{m_{\rho,\tau}^2}\psi(m_{\rho,\tau}z+w)-\frac{1}{n_{\rho,\tau}^2}\psi(n_{\rho,\tau}z+w)\biggr],\\ (1-\rho)zw-\tau w^2+&\frac{\sqrt{\rho+\tau-1}+\sqrt{\rho\tau}}{2}\varphi(m_{\rho,\tau}z+w)\\ +&\frac{\sqrt{\rho+\tau-1}-\sqrt{\rho\tau}}{2}\varphi(n_{\rho,\tau}z+w)\\ +&\,\psi(m_{\rho,\tau}z+w)-\psi(n_{\rho,\tau}z+w)\Biggr)\,,\end{aligned}$$

where the square roots of $\rho\tau$ and of $\rho+\tau-1$ are chosen so that their product is equal to the previously chosen square root of $\rho\tau(\rho+\tau-1)$, and φ, $\psi\in\mathbb{C}[\![\zeta]\!]$ are power series of order at least 3.

3. Germs tangent to the identity

In this section we shall assume $n=\mu=2$ and $\Lambda=I$, that is we shall be interested in 2-dimensional germs tangent to the identity of order 2. We shall keep using the notations introduced in the previous section. It should be recall that in his monumental work [É1] (see [É2] for a survey) Écalle studied the formal classification of germs tangent to the identity in dimension n, giving a complete set of formal invariants for germs satisfying a generic condition: the existence of at least one non-degenerate characteristic direction (an eigenradius, in Écalle's terminology). A *characteristic direction* of a germ tangent to the identity F is a non-zero direction v such that $F_\mu(v)=\lambda v$ for some $\lambda\in\mathbb{C}$, where F_μ is the first (nonlinear) non-vanishing term in the homogeneous expansion of F. The characteristic direction v is *degenerate* if $\lambda=0$.

For this reason, we decided to discuss here the cases *without* non-degenerate characteristic directions, that is the cases (1_{00}), (1_{10}) and (2_{001}), that cannot be dealt with Écalle's methods. Furthermore, we shall also study the somewhat special case (∞), where all directions are characteristic; and we shall examine in detail case $(2_{10\rho})$, where interesting second-order resonance phenomena appear.

When $\Lambda=I$ the operator $L=L_{F_2,\Lambda}$ is given by

$$L(H)=\mathrm{Jac}(H)\cdot F_2-\mathrm{Jac}(F_2)\cdot H\,.$$

In particular, $L(F_2)=O$ always; therefore we cannot apply Proposition 1.7 (nor other similar conditions stated in [WZP2]), and we shall compute the second order normal form only.

• *Case* (∞).
In this case we have

$$L(u_{d,j})=(d-2)u_{d+1,j+1}-v_{d+1,j}\qquad\text{and}\qquad L(v_{d,j})=(d-1)v_{d+1,j+1}$$

for all $d \geq 2$ and $j = 0, \ldots, d$. Therefore

$$\operatorname{Im} L|_{\mathcal{H}^d} = \begin{cases} \operatorname{Span}\left(u_{d+1,2}, \ldots, u_{d+1,d+1}, (d-2)u_{d+1,1} - v_{d+1,0}, v_{d+1,1}, \ldots, v_{d+1,d+1}\right) & \text{for } d > 2, \\ \operatorname{Span}\left(v_{3,0}, \ldots, v_{3,3}\right) & \text{for } d = 2. \end{cases}$$

Thus

$$(\operatorname{Im} L|_{\mathcal{H}^d})^\perp = \begin{cases} \operatorname{Span}\left(u_{d+1,0}, (d+1)u_{d+1,1} + (d-2)v_{d+1,0}\right) & \text{for } d > 2, \\ \operatorname{Span}\left(u_{3,0}, \ldots, u_{3,3}\right) & \text{for } d = 2. \end{cases}$$

It then follows that every formal power series of the form

$$F(z,w) = (z + z^2 + O_3, w + zw + O_3)$$

has as second order normal form

$$G(z,w) = \bigl(z + z^2 + a_0 z^3 + a_1 z^2 w + a_2 z w^2 + \varphi(w) + z\psi'(w), zw + w\psi'(w) - 3\psi(w)\bigr)$$

where $\varphi \in \mathbb{C}[\![\zeta]\!]$ is a power series of order at least 3, $\psi \in \mathbb{C}[\![\zeta]\!]$ is a power series of order at least 4 and a_0, a_1, $a_2 \in \mathbb{C}$.

• *Case* (1_{00}).
In this case we have

$$L(u_{d,j}) = (j-d)u_{d+1,j+2} + 2v_{d+1,j+1} \qquad \text{and} \qquad L(v_{d,j}) = (j-d)v_{d+1,j+2}$$

for all $d \geq 2$ and $j = 0, \ldots, d$. Therefore

$$\operatorname{Im} L|_{\mathcal{H}^d} = \operatorname{Span}\left(2v_{d+1,1} - du_{d+1,2}, u_{d+1,3}, \ldots, u_{d+1,d+1}, v_{d+1,2}, \ldots, v_{d+1,d+1}\right) ,$$

and thus

$$(\operatorname{Im} L|_{\mathcal{H}^d})^\perp = \operatorname{Span}\left(u_{d+1,0}, u_{d+1,1}, v_{d+1,0}, u_{d+1,2} + v_{d+1,1}\right) .$$

It then follows that every formal power series of the form

$$F(z,w) = (z + O_3, w - z^2 + O_3)$$

has as second order normal form

$$G(z,w) = \bigl(z + w\varphi_1(w) + z\varphi_2(w) + z^2\psi(w), w - z^2 + w\varphi_3(w) + zw\psi(w)\bigr) ,$$

where φ_1, φ_2, $\varphi_3 \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 2, and $\psi \in \mathbb{C}[\![\zeta]\!]$ is a power series of order at least 1.

• *Case* (1_{10}).
In this case we have

$$L(u_{d,j}) = (2-d)u_{d+1,j+1} - (d-j)u_{d+1,j+2} + 2v_{d+1,j+1} + v_{d+1,j}$$

and

$$L(v_{d,j}) = (1-d)v_{d+1,j+1} - (d-j)v_{d+1,j+2}$$

for all $d \geq 2$ and $j = 0, \ldots, d$. Therefore

$$\operatorname{Im} L|_{\mathcal{H}^d} = \begin{cases} \operatorname{Span}\left((2-d)u_{d+1,1} + v_{d+1,0}, u_{d+1,2}, \ldots, u_{d+1,d+1}, v_{d+1,1}, \ldots, v_{d+1,d+1}\right) & \text{for } d > 2, \\ \operatorname{Span}\left(v_{3,0} - 2u_{3,2}, u_{3,3}, v_{3,1}, v_{3,2}, v_{3,3}\right) & \text{for } d = 2, \end{cases}$$

and thus

$$(\operatorname{Im} L|_{\mathcal{H}^d})^\perp = \begin{cases} \operatorname{Span}\left(u_{d+1,0}, (d+1)u_{d+1,1} + (d-2)v_{d+1,0}\right) & \text{for } d > 2, \\ \operatorname{Span}\left(u_{3,0}, u_{3,1}, 3u_{3,2} + 2v_{3,0}\right) & \text{for } d = 2. \end{cases}$$

It then follows that every formal power series of the form

$$F(z,w) = (z - z^2 + O_3, w - z^2 - zw + O_3)$$

has as second order normal form

$$G(z,w) = \bigl(z - z^2 + \varphi(w) + a_1 z w^2 + 3a_2 z^2 w + z\psi'(w), w - z^2 - zw + 2a_2 w^3 + w\psi'(w) - 3\psi(w)\bigr) ,$$

where $\varphi \in \mathbb{C}[\![\zeta]\!]$ is a power series of order at least 3, $\psi \in \mathbb{C}[\![\zeta]\!]$ is a power series of order at least 4, and a_1, $a_2 \in \mathbb{C}$.

• *Case* (2_{001}).
In this case we have

$$L(u_{d,j}) = (d-j)u_{d+1,j+1} - v_{d+1,j} \qquad \text{and} \qquad L(v_{d,j}) = (d-j-1)v_{d+1,j+1}$$

for all $d \geq 2$ and $j = 0, \ldots, d$. It follows that

$$\operatorname{Im} L|_{\mathcal{H}^d} = \operatorname{Span}\left(du_{d+1,1} - v_{d+1,0}, u_{d+1,2}, \ldots, u_{d+1,d}, v_{d+1,1}, \ldots, v_{d+1,d+1}\right)$$

and hence

$$(\operatorname{Im} L|_{\mathcal{H}^d})^{\perp} = \operatorname{Span}\left(u_{d+1,0}, u_{d+1,d+1}, (d+1)u_{d+1,1} + dv_{d+1,0}\right) .$$

It then follows that every formal germ of the form

$$F(z,w) = (z + O_3, w + zw + O_3)$$

has as second order normal form

$$G(z,w) = \big(z + \varphi_1(z) + \varphi_2(w) + z\psi'(w), zw + w\psi'(w) - \psi(w)\big)$$

where φ_1, φ_2, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3.

• *Case* $(2_{10\rho})$.
In this case we have

$$L(u_{d,j}) = (d-j-d\rho+2\rho)u_{d+1,j+1} + (\rho-1)v_{d+1,j} \quad \text{and} \quad L(v_{d,j}) = (d-j-d\rho+\rho-1)v_{d+1,j+1} \tag{3.1}$$

for all $d \geq 2$ and $j = 0, \ldots, d$. Here we shall see the resonance phenomena we mentioned at the beginning of this section: for some values of ρ the dimension of the kernel of $L|_{\mathcal{H}^d}$ increases, and in some cases we shall end up with a normal form depending on power series evaluated in monomials of the form $z^{b-a}w^a$.

Let us put

$$E_d = \left\{ \frac{d-j-1}{d-1} \;\middle|\; j = 0, \ldots, d \right\} \setminus \{0\} \quad \text{and} \quad F_d = \left\{ \frac{d-j}{d-2} \;\middle|\; j = 0, \ldots, d-1 \right\}$$

(we are excluding 0 because $\rho \neq 0$ by assumption), where E_d is defined for all $d \geq 2$ whereas F_d is defined for all $d \geq 3$, and set

$$\mathcal{E} = \bigcup_{d \geq 2} E_d = \big((0,1] \cap \mathbb{Q}\big) \cup \left\{ -\frac{1}{n} \;\middle|\; n \in \mathbb{N}^* \right\}$$

and

$$\mathcal{F} = \bigcup_{d \geq 3} F_d = \big((0,1] \cap \mathbb{Q}\big) \cup \left\{ 1 + \frac{1}{n}, 1 + \frac{2}{n} \;\middle|\; n \in \mathbb{N}^* \right\} .$$

So $\mathcal{E}$ is the set of $\rho \in \mathbb{C}^*$ such that $L(v_{d,j}) = 0$ for some $d \geq 2$ and $0 \leq j \leq d$, while $\mathcal{F}$ is the set of $\rho \in \mathbb{C}^*$ such that $L(u_{d,j}) = (\rho - 1)v_{d+1,j}$ for some $d \geq 3$ and $0 \leq j \leq d-1$.

Let us first discuss the non-resonant case, when $\rho \notin \mathcal{E} \cup \mathcal{F}$. Then none of the coefficients in (3.1) vanishes, and thus

$$\operatorname{Im} L|_{\mathcal{H}^2} = \operatorname{Span}\left(2u_{3,1} + (\rho-1)v_{3,0}, u_{3,2}, v_{3,1}, v_{3,2}, v_{3,3}\right)$$

and

$$\operatorname{Im} L|_{\mathcal{H}^d} = \operatorname{Span}\left((d - d\rho + 2\rho)u_{d+1,1} + (\rho - 1)v_{d+1,0}, u_{d+1,2}, \ldots, u_{d+1,d+1}, v_{d+1,1}, \ldots, v_{d+1,d+1}\right) ,$$

for $d \geq 3$, and hence

$$(\operatorname{Im} L|_{\mathcal{H}^d})^\perp = \begin{cases} \operatorname{Span}\left(u_{d+1,0}, (1-\rho)(d+1)u_{d+1,1} + (d(1-\rho)+2\rho)v_{d+1,0}\right) & \text{for } d \geq 3, \\ \operatorname{Span}\left(u_{3,0}, u_{3,3}, 3(1-\rho)u_{3,1} + 2v_{3,0}\right) & \text{for } d = 2. \end{cases}$$

It then follows that every formal germ of the form

$$F(z,w) = (z - \rho z^2 + O_3, w + (1-\rho)zw + O_3)$$

with $\rho \notin \mathcal{E} \cup \mathcal{F}$ (and $\rho \neq 0$) has as second order normal form

$$G(z,w) = \big(z - \rho z^2 + az^3 + \varphi(w) + (1-\rho)z\psi'(w), w + (1-\rho)zw + (1-\rho)w\psi'(w) + (3\rho - 1)\psi(z)\big) ,$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3, and $a \in \mathbb{C}$.

Assume now $\rho \in \mathcal{F} \setminus \mathcal{E}$. Then $L(v_{d,j}) \neq O$ always, and thus $v_{d+1,j} \in \operatorname{Im} L|_{\mathcal{H}^d}$ for all $d \geq 2$ and all $j = 1, \ldots, d+1$. Since $\rho > 1$, if $d > 2$ it also follows that $u_{d+1,j+1} \in \operatorname{Im} L|_{\mathcal{H}^d}$ for $j = 1, \ldots, d$.

Now, if $\rho = 1 + (1/n)$ then

$$\frac{d}{d-2} = \rho \iff d = 2(n+1) ,$$

and

$$\frac{d-1}{d-2} = \rho \iff d = n + 2 .$$

Taking care of the case $d = 2$ separately, we then have

$$\begin{aligned} &\operatorname{Im} L|_{\mathcal{H}^d} \\ &= \begin{cases} \operatorname{Span}\left((d - d\rho + 2\rho)u_{d+1,1} + (\rho - 1)v_{d+1,0}, u_{d+1,2}, \ldots, u_{d+1,d+1}, v_{d+1,1}, \ldots, v_{d+1,d+1}\right) \\ \hfill \text{for } d \geq 3,\ d \neq n+2,\ 2(n+1), \\ \operatorname{Span}\left(u_{d+1,1} + (\rho - 1)v_{d+1,0}, u_{d+1,3}, \ldots, u_{d+1,d+1}, v_{d+1,1}, \ldots, v_{d+1,d+1}\right) \quad \text{for } d = n+2, \\ \operatorname{Span}\left(u_{d+1,2}, \ldots, u_{d+1,d+1}, v_{d+1,0}, \ldots, v_{d+1,d+1}\right) \quad \text{for } d = 2(n+1), \\ \operatorname{Span}\left(2u_{3,1} + (\rho - 1)v_{3,0}, u_{3,2}, v_{3,1}, v_{3,2}, v_{3,3}\right) \quad \text{for } d = 2, \end{cases} \end{aligned}$$

and hence

$$\begin{aligned} &(\operatorname{Im} L|_{\mathcal{H}^d})^\perp \\ &= \begin{cases} \operatorname{Span}\left(u_{d+1,0}, (1-\rho)(d+1)u_{d+1,1} + (d(1-\rho)+2\rho)v_{d+1,0}\right) & \text{for } d \geq 3,\ d \neq n+2,\ 2(n+1), \\ \operatorname{Span}\left(u_{d+1,0}, u_{d+1,2}, (1-\rho)(d+1)u_{d+1,1} + v_{d+1,0}\right) & \text{for } d = n+2, \\ \operatorname{Span}\left(u_{d+1,0}, u_{d+1,1}\right) & \text{for } d = 2(n+1), \\ \operatorname{Span}\left(u_{3,0}, u_{3,3}, 3(1-\rho)u_{3,1} + 2v_{3,0}\right) & \text{for } d = 2. \end{cases} \end{aligned}$$

It then follows that every formal germ of the form

$$F(z,w) = \left(z - \left(1 + \frac{1}{n}\right) z^2 + O_3, w - \frac{1}{n} zw + O_3\right)$$

with $n \in \mathbb{N}^*$ has as second order normal form

$$G(z,w) = \left(z - \left(1 + \frac{1}{n}\right) z^2 + \varphi(w) + (1-\rho)z\psi'(w) + a_0 z^3 + a_1 z^2 w^{n+1}, \right.$$
$$\left. w - \frac{1}{n} zw + (1-\rho)w\psi'(w) + (3\rho - 1)\psi(w)\right) ,$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3, and a_0, $a_1 \in \mathbb{C}$.

If instead $\rho = 1 + (2/m)$ with m odd (if m is even we are again in the previous case) then

$$\frac{d}{d-2} = \rho \quad \Longleftrightarrow \quad d = m + 2 ,$$

whereas $\frac{d-1}{d-2} \neq \rho$ always. Hence

$$\operatorname{Im} L|_{\mathcal{H}^d} = \begin{cases} \operatorname{Span}\left((d - d\rho + 2\rho)u_{d+1,1} + (\rho-1)v_{d+1,0}, u_{d+1,2}, \ldots, u_{d+1,d+1}, v_{d+1,1}, \ldots, v_{d+1,d+1}\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } d \geq 3,\ d \neq m+2, \\ \operatorname{Span}\left(u_{d+1,2}, \ldots, u_{d+1,d+1}, v_{d+1,0}, \ldots, v_{d+1,d+1}\right) \qquad \text{for } d = m+2, \\ \operatorname{Span}\left(2u_{3,1} + (\rho-1)v_{3,0}, u_{3,2}, v_{3,1}, v_{3,2}, v_{3,3}\right) \qquad \text{for } d = 2, \end{cases}$$

and thus

$$(\operatorname{Im} L|_{\mathcal{H}^d})^\perp = \begin{cases} \operatorname{Span}\left(u_{d+1,0}, (1-\rho)(d+1)u_{d+1,1} + (d - d\rho + 2\rho)v_{d+1,0}\right) & \text{for } d \geq 3,\ d \neq m+2, \\ \operatorname{Span}\left(u_{d+1,0}, u_{d+1,1}\right) & \text{for } d = m+2, \\ \operatorname{Span}\left(u_{3,0}, u_{3,3}, 3(1-\rho)u_{3,1} + 2v_{3,0}\right) & \text{for } d = 2. \end{cases}$$

It then follows that every formal germ of the form

$$F(z,w) = \left(z - \left(1 + \frac{2}{m}\right) z^2 + O_3, w - \frac{2}{m} zw + O_3\right)$$

with $m \in \mathbb{N}^*$ odd has as second order normal form

$$G(z,w) = \left(z - \left(1 + \frac{2}{m}\right) z^2 + \varphi(w) + a_0 z^3 + (1-\rho)z(w\psi'(w) + \psi(w)), \right.$$
$$\left. w - \frac{2}{m} zw + (1-\rho)w^2\psi'(w) + 2\rho w\psi(w)\right) ,$$

where $\varphi \in \mathbb{C}[\![\zeta]\!]$ is a power series of order at least 3, $\psi \in \mathbb{C}[\![\zeta]\!]$ is a power series of order at least 2, and $a_0 \in \mathbb{C}$.

Now let us consider the case $\rho = -1/n \in \mathcal{E} \setminus \mathcal{F}$. In this case the coefficients in the expression of $L(u_{d,j})$ are always different from zero (with the exception of $d = j = 2$), whereas

$$d - j - d\rho + \rho - 1 = 0 \quad \Longleftrightarrow \quad j = d = n + 1 .$$

It follows that

$$\operatorname{Im} L|_{\mathcal{H}^d} = \begin{cases} \operatorname{Span}\left((d - d\rho + 2\rho)u_{d+1,1} + (\rho-1)v_{d+1,0}, u_{d+1,2}, \ldots, u_{d+1,d+1}, v_{d+1,1}, \ldots, v_{d+1,d+1}\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } d \geq 3,\ d \neq n+1, \\ \operatorname{Span}\left((d - d\rho + 2\rho)u_{d+1,1} + (\rho-1)v_{d+1,0}, u_{d+1,2}, \ldots, u_{d+1,d+1}, v_{d+1,1}, \ldots, v_{d+1,d}\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } d = n+1, \\ \operatorname{Span}\left(2u_{3,1} + (\rho-1)v_{3,0}, u_{3,2}, v_{3,1}, v_{3,2}, v_{3,3}\right) \qquad \text{for } d = 2, \end{cases}$$

and thus

$$
\begin{aligned}
&(\operatorname{Im} L|_{\mathcal{H}^d})^\perp \\
&= \begin{cases} \operatorname{Span}\left(u_{d+1,0}, (1-\rho)(d+1)u_{d+1,1} + (d - d\rho + 2\rho)v_{d+1,0}\right) & \text{for } d \geq 3,\ d \neq n+1, \\ \operatorname{Span}\left(u_{d+1,0}, v_{d+1,d+1}, (1-\rho)(d+1)u_{d+1,1} + (d - d\rho + 2\rho)v_{d+1,0}\right) & \text{for } d = n+1, \\ \operatorname{Span}\left(u_{3,0}, u_{3,3}, 3(1-\rho)u_{3,1} + 2v_{3,0}\right) & \text{for } d = 2. \end{cases}
\end{aligned}
$$

It then follows that every formal germ of the form

$$
F(z,w) = \left(z + \frac{1}{n}z^2 + O_3, w + \left(1 + \frac{1}{n}\right)zw + O_3\right)
$$

with $n \in \mathbb{N}^*$ has as second order normal form

$$
\begin{aligned}
G(z,w) = \Bigg(&z + \frac{1}{n}z^2 + \varphi(w) + a_0 z^3 + (1-\rho)z(w\psi'(w) + \psi(w)), \\
&w + \left(1 + \frac{1}{n}\right)zw + \psi(z) + a_1 z^{n+2} + (1-\rho)w^2\psi'(w) + 2\rho w\psi(w)\Bigg)\ ,
\end{aligned}
$$

where $\varphi \in \mathbb{C}[\![\zeta]\!]$ is a power series of order at least 3, $\psi \in \mathbb{C}[\![\zeta]\!]$ is a power series of order at least 2, and $a_0,\ a_1 \in \mathbb{C}$.

Let us now discuss the extreme case $\rho = 1$. It is clear that

$$
\operatorname{Im} L|_{\mathcal{H}^d} = \operatorname{Span}\left(u_{d+1,1}, u_{d+1,2}, u_{d+1,4}, \ldots, u_{d+1,d+1}, v_{d+1,2}, \ldots, v_{d+1,d+1}\right)\ ,
$$

and hence

$$
(\operatorname{Im} L|_{\mathcal{H}^d})^\perp = \operatorname{Span}\left(u_{d+1,0}, u_{d+1,3}, v_{d+1,0}, v_{d+1,1}\right)\ ,
$$

It then follows that every formal germ of the form

$$
F(z,w) = \left(z - z^2 + O_3, w + O_3\right)
$$

has as second order normal form

$$
G(z,w) = \left(z - z^2 + \varphi_1(w) + z^3\psi(w), w + \varphi_2(w) + z\varphi_3(w)\right)\ ,
$$

where $\varphi_1,\ \varphi_2 \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3, $\varphi_3 \in \mathbb{C}[\![\zeta]\!]$ is a power series of order at least 2, and $\psi \in \mathbb{C}[\![\zeta]\!]$ is a power series.

We are left with the case $\rho \in (0,1) \cap \mathbb{Q}$. Write $\rho = a/b$ with $a,\ b \in \mathbb{N}$ coprime and $0 < a < b$. Now

$$
d - j - 1 - \frac{a}{b}(d-1) = 0 \quad \Longleftrightarrow \quad j = \frac{(d-1)(b-a)}{b}\ ;
$$

since a and b are coprime, this happens if and only if $d = b\ell + 1$ and $j = (b-a)\ell$ for some $\ell \geq 1$. Analogously,

$$
d - j - \frac{a}{b}(d-2) = 0 \quad \Longleftrightarrow \quad j = d - \frac{a(d-2)}{b}\ ;
$$

again, being a and b coprime, this happens if and only if $d = b\ell + 2$ and $j = (b-a)\ell + 2$ for some $\ell \geq 0$. It follows that

$$
\operatorname{Im} L|_{\mathcal{H}^d} = \begin{cases}
\operatorname{Span}\big((d - d\rho + 2\rho)u_{d+1,1} + (\rho-1)v_{d+1,0}, u_{d+1,2}, \ldots, u_{d+1,d+1}, v_{d+1,1}, \ldots, v_{d+1,d+1}\big) \\
\hfill \text{for } d \geq 3,\ d \not\equiv 1,\ 2 \pmod b) \\
\operatorname{Span}\big((d - d\rho + 2\rho)u_{d+1,1} + (\rho-1)v_{d+1,0}, u_{d+1,2}, \ldots, \widehat{u_{d+1,(b-a)\ell+2}}, \ldots, u_{d+1,d+1}, \\
\quad v_{d+1,1}, \ldots, \widehat{v_{d+1,(b-a)\ell+1}}, \ldots, v_{d+1,d+1}, \frac{a}{b}u_{d+1,(b-a)\ell+2} - \left(\frac{a}{b} - 1\right) v_{d+1,(b-a)\ell+1}\big) \\
\hfill \text{for } d = b\ell + 1, \\
\operatorname{Span}\big((d - d\rho + 2\rho)u_{d+1,1} + (\rho-1)v_{d+1,0}, u_{d+1,2}, \ldots, \widehat{u_{d+1,(b-a)\ell+3}}, \ldots, u_{d+1,d+1}, \\
\quad v_{d+1,1}, \ldots, v_{d+1,d+1}\big) \hfill \text{for } d = b\ell + 2, \\
\operatorname{Span}\big(2u_{3,1} + (\rho-1)v_{3,0}, u_{3,2}, v_{3,1}, v_{3,2}, v_{3,3}\big) \hfill \text{for } d = 2,
\end{cases}
$$

(where the hat indicates that that term is missing from the list), and thus

$$
(\operatorname{Im} L|_{\mathcal{H}^d})^\perp = \begin{cases}
\operatorname{Span}\big(u_{d+1,0}, (1-\rho)u_{d+1,1} + (d - d\rho + 2\rho)v_{d+1,0}\big) & \text{for } d \geq 3,\ d \not\equiv 1,\ 2 \pmod b), \\
\operatorname{Span}\big(u_{d+1,0}, (1-\rho)u_{d+1,1} + (d - d\rho + 2\rho)v_{d+1,0}, \\
\quad (b-a)(a\ell+1)u_{d+1,(b-a)\ell+2} + a\big((b-a)\ell+2\big)v_{d+1,(b-a)\ell+1}\big) & \text{for } d = b\ell + 1, \\
\operatorname{Span}\big(u_{d+1,0}, u_{d+1,(b-a)\ell+3}, (1-\rho)u_{d+1,1} + (d - d\rho + 2\rho)v_{d+1,0}\big) & \text{for } d = b\ell + 2, \\
\operatorname{Span}\big(u_{3,0}, u_{3,3}, 3(1-\rho)u_{3,1} + 2v_{3,0}\big) & \text{for } d = 2.
\end{cases}
$$

It then follows that every formal germ of the form

$$
F(z,w) = \left(z - \frac{a}{b}z^2 + O_3, w + \left(1 - \frac{a}{b}\right) zw + O_3\right)
$$

with $a/b \in (0,1) \cap \mathbb{Q}$ and a, b coprime, has as second order normal form

$$
\begin{aligned}
G(z,w) &= \bigg(z - \frac{a}{b}z^2 + \varphi(w) + z^3\varphi_0(z^{b-a}w^a) + (b-a)\frac{\partial}{\partial w}\big(z^2 w\chi(z^{b-a}w^a)\big) + \left(1 - \frac{a}{b}\right) z\big(w\psi'(w) + \psi(w)\big), \\
&\qquad w + \left(1 - \frac{a}{b}\right) zw + a\frac{\partial}{\partial z}\big(z^2 w\chi(z^{b-a}w^a)\big) + \left(1 - \frac{a}{b}\right) w^2\psi'(w) + 2\frac{a}{b}w\psi(w)\bigg)\ ,
\end{aligned}
$$

where φ, $\psi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 3, and φ_0, $\chi \in \mathbb{C}[\![\zeta]\!]$ are power series of order at least 1.

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